Leonid Shaikhet

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ISBN 978-0-85729-684-9 e-ISBN 978-0-85729-685-6
DOI 10.1007/978-0-85729-685-6
Springer London Dordrecht Heidelberg New York

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2011930099

Cover design: VTeX UAB, Lithuania

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Hereditary systems (or systems with delays, or systems with aftereffect) are systems, which future development depends not only on their present state but also on their previous history. Systems of such type are widely used to model processes in physics, mechanics, automatic regulation, economy, biology, ecology etc. (see, e.g., [6, 8, 9, 11, 25, 92, 97, 119, 120, 122, 123, 131, 152, 178, 189, 210, 263]). An important element in the study of such systems is their stability. As it was proposed by Krasovskii [149–151], stability condition for differential equation with delays can be obtained using appropriate Lyapunov functional. By that the construction of different Lyapunov functionals for one differential equation allows to get different stability conditions for the solution of this equation. However the construction of each Lyapunov functional required a unique work from its own author. In 1975, Shaikhet [226] introduced a parametric family of Lyapunov functionals, so that an infinite number of Lyapunov functionals were used simultaneously. This way allowed to get different stability conditions for considered equation using only one Lyapunov functional. At last in the 1990s the general method of Lyapunov functionals construction was proposed by Kolmanovskii and Shaikhet for stochastic functional-differential equations and developed later consistently for stochastic difference equations with discrete time and continuous time, for partial differential equations [40, 125–130, 132–140, 177, 182, 194–199, 228–231, 233, 234, 236, 238–241, 244, 245]. The method was successfully applied to stability research of some mathematical models in mechanics and biology [24, 28–30, 235, 242, 246, 247].

Stability theory for stochastic differential equations (both without delays and with delays) is well studied (see, e.g., [13, 14, 82, 87–89, 97, 109, 112, 119, 120, 122, 123, 180, 181, 189]). Difference equations, which arise as numerical analogues of differential or integral equations as well as independent mathematical models of dynamical systems with discrete time, have also enjoyed a considerable share of research attention [5–9, 12, 15, 50–54, 58, 62, 66–73, 96, 99, 111, 115–118, 121, 124, 161, 162, 186, 207, 211, 212, 214, 251, 270, 276].

In this book, consisting of ten chapters, the general method of Lyapunov functionals construction for stochastic difference Volterra equations is expounded.

Note that the term "Lyapunov functional" is used here not as a "function of functions" only but in some more wide sense as a "function of trajectories" or as a "function of solutions of equations". Taking into account that our general method of Lyapunov functionals construction can be applied not for difference equations with discrete time only but also for differential equations and for difference equations with continuous time the use here the term "Lyapunov functional" is defensible.

Introductory Chap. 1 presents basic definitions, Lyapunov type theorems, formal procedure of Lyapunov functionals construction and some useful lemmas.

In Chap. 2 the procedure of Lyapunov functionals construction, described in Chap. 1, is used for p-stability investigation of a simple stochastic difference equation with constant coefficients. It is shown that different ways of Lyapunov functionals construction allow to get different conditions for asymptotic p-stability of the zero solution of this equation.

Chapter 3 generalizes the material from Chap. 2, by applying the procedure of Lyapunov functionals construction to obtain conditions for stability of stochastic difference equation with stationary coefficients. Four different ways of Lyapunov functionals construction are shown. In addition, asymptotic behavior of the solution is studied by characteristic equation.

In Chap. 4 via the procedure of Lyapunov functionals construction different types of sufficient stability conditions are obtained for linear equations with nonstationary coefficients.

In Chap. 5 some important features of the proposed method of Lyapunov functionals construction are considered. In particular, the necessary and sufficient condition for asymptotic mean square stability of the zero solution of stochastic linear difference equation and sufficient stability conditions for the difference equation with Markovian switching are obtained.

In Chap. 6 the general method of Lyapunov functionals construction is used to get asymptotic mean square stability conditions for systems of stochastic linear difference equations with varying delays. Sufficient stability conditions are formulated in terms of existence of positive definite solutions of some matrix Riccati equations.

In Chap. 7 the procedure of Lyapunov functionals construction considered above is applied to different types of nonlinear stochastic difference equations and to different types of stability: asymptotic mean square stability, stability in probability, almost sure stability.

Chapter 8 studies asymptotic behavior of solutions of stochastic difference Volterra equations of the second kind. For that the general method of Lyapunov functionals construction is used as well as the resolvent representation of solutions. In particular, linear and nonlinear equations are considered with constant and with variable coefficients.

In Chap. 9 it is shown that after some modification of the basic Lyapunov type theorems the general method of Lyapunov functionals construction can be applied also for difference equations with continuous time that are enough popular with researches. Some of the previous results are reformulated for such equations and some peculiarities of its investigation are shown.

In Chap. 10 a capability of difference analogues of differential equations to save a property of stability of solutions of considered differential equations is discussed. In particular, sufficient conditions on discretization step, at which a property of stability is saved, are obtained for some known mathematical models: inverted pendulum, Nicholson's blowflies equation, predator–prey model, for integro-differential equation of convolution type.

The bibliography at the end of the book does not pretend to be complete and includes some of the author's publications [226–247], his publications jointly with coauthors [11, 24, 28–30, 40, 60, 80, 125–141, 148, 153, 177, 182, 194–199, 215, 248–250] as well as the literature used by the author during his preparation of this book.

The book is addressed both to experts in stability theory as well as to a wider audience of professionals and students in pure and computational mathematics, physics, engineering, biology and so on.

The author will appreciate receiving useful remarks, comments and suggestions.

The book is mostly based on the results obtained by the author independently or jointly with coauthors, in particular, with the friend and colleague V. Kolmanovskii, with whom the author is glad and happy to collaborate for more than 30 years.

Donetsk, Ukraine Leonid Shaikhet

Contents

Chapter 1
Lyapunov-type Theorems and Procedure of Lyapunov Functionals Construction

1.1 Definitions and Basic Lyapunov-type Theorem

Let h be a given nonnegative integer or $h=\infty$, i be a discrete time, $i \in Z \cup Z_0$, $Z=\{0,1,\ldots\}$, $Z_0=\{-h,\ldots,0\}$, S be a space of sequences with elements in $\mathbf{R}^n$.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathfrak{F}_i \in \mathfrak{F}$, $i \in Z$, be a family of σ-algebras, $\mathbf{E}$ be an expectation, ξ_j, $j \in Z$, be a sequence of $\mathfrak{F}_j$-adapted random variables, and process $x_i \in \mathbf{R}^n$ be a solution of the equation

$$x_{i+1} = F(i, x_{-h}, \ldots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \ldots, x_j)\xi_{j+1}, \quad i \in Z, \tag{1.1}$$

with the initial condition

$$x_i = \varphi_i, \quad i \in Z_0. \tag{1.2}$$

Here $F : Z \times S \to \mathbf{R}^n$, $G : Z \times Z \times S \to \mathbf{R}^n$. It is assumed that $F(i,\ldots)$ does not depend on x_j by $j > i$, $G(i,j,\ldots)$ does not depend on x_k by $k > j$ and $F(i,0,\ldots,0)=0$, $G(i,j,0,\ldots,0)=0$.

For arbitrary functional $V_i = V(i, x_{-h}, \ldots, x_i)$, $i \in Z$, the operator ΔV_i is defined by

$$\Delta V_i = V(i+1, x_{-h}, \ldots, x_{i+1}) - V(i, x_{-h}, \ldots, x_i). \tag{1.3}$$

Definition 1.1 The solution of (1.1) with initial function (1.2) for some $p > 0$ is called:

- Uniformly p-bounded if $\sup_{i \in Z} \mathbf{E}|x_i|^p < \infty$.
- Asymptotically p-trivial if $\lim_{i \to \infty} \mathbf{E}|x_i|^p = 0$.
- p-summable if $\sum_{i=0}^{\infty} \mathbf{E}|x_i|^p < \infty$.

Definition 1.2 The trivial solution of (1.1) for some $p > 0$ is called:

- p-stable, $p > 0$, if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^p < \epsilon$, $i \in Z$, if $\|\varphi\|^p = \sup_{i \in Z_0} \mathbf{E}|\varphi_i|^p < \delta$.

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_1,

- Asymptotically p-stable if it is p-stable and for each initial function φ_i the solution x_i of (1.1) is asymptotically p-trivial.

In particular, if $p=2$, then the solution of (1.1) is called mean square bounded, mean square stable, asymptotically mean square stable and so on.

Remark 1.1 Note that if the solution of (1.1) is p-summable, then it is p-bounded and asymptotically p-trivial.

Theorem 1.1 *Let there exist a nonnegative functional $V_i=V(i,x_{-h},\dots,x_i)$ which satisfies the conditions*

$$\mathbf{E}V(0,\varphi_{-h},\dots,\varphi_0)\le c_1\|\varphi\|^p, \tag{1.4}$$

$$\mathbf{E}\Delta V_i\le -c_2\mathbf{E}|x_i|^p,\quad i\in Z, \tag{1.5}$$

where c_1, c_2 and p are positive constants. Then the trivial solution of (1.1) *is asymptotically p-stable.*

Proof From (1.3), (1.5) it follows that

$$\sum_{j=0}^{i}\mathbf{E}\Delta V_j=\mathbf{E}V(i+1,x_{-h},\dots,x_{i+1})-\mathbf{E}V(0,x_{-h},\dots,x_0)\le -c_2\sum_{j=0}^{i}\mathbf{E}|x_j|^p.$$

From this and (1.4) we obtain

$$\mathbf{E}|x_i|^p\le\sum_{j=0}^{i}\mathbf{E}|x_j|^p\le\frac{1}{c_2}\mathbf{E}V(0,x_{-h},\dots,x_0)\le\frac{c_1}{c_2}\|\varphi\|^p, \tag{1.6}$$

i.e. the trivial solution of (1.1) is p-stable.

From (1.6) it follows also that the solution of (1.1) is p-summable for each initial function φ such that $\|\varphi\|<\infty$ and therefore (Remark 1.1) it is asymptotically p-trivial. So, the trivial solution of (1.1) is asymptotically p-stable. Theorem 1.1 is proven. □

Corollary 1.1 *Let there exist a nonnegative functional $V_i=V(i,x_{-h},\dots,x_i)$, which satisfies conditions* (1.4) *and*

$$\mathbf{E}\Delta V_i=-c\mathbf{E}|x_i|^p,\quad i\in Z.$$

Then the inequality $c>0$ is a necessary and sufficient condition for asymptotic p-stability of the trivial solution of (1.1).

Proof Sufficiency follows from Theorem 1.1. To prove a necessity it is enough to show that if $c \le 0$ then via (1.3)

$$\sum_{j=0}^{i-1} \mathbf{E}\Delta V_j = \mathbf{E}V_i - \mathbf{E}V_0 \ge 0$$

or $\mathbf{E}V_i \ge \mathbf{E}V_0 > 0$ for each initial function $\varphi \not\equiv 0$. It means that the trivial solution of (1.1) cannot be asymptotically p-stable. □

1.2 Formal Procedure of Lyapunov Functionals Construction

From Theorem 1.1 it follows that the stability investigation of stochastic difference equations can be reduced to construction of appropriate Lyapunov functionals.

Below, a formal procedure for the construction of Lyapunov functionals for (1.1), (1.2) is proposed. This procedure consists of four steps.

Step 1 Represent the functions F and G on the right-hand side of (1.1) in the form

$$\begin{aligned}
F(i, x_{-h}, \dots, x_i) &= F_1(i, x_{i-\tau}, \dots, x_i) \\
&\quad + F_2(i, x_{-h}, \dots, x_i) + \Delta F_3(i, x_{-h}, \dots, x_i), \\
F_1(i, 0, \dots, 0) &\equiv F_2(i, 0, \dots, 0) \equiv F_3(i, 0, \dots, 0) \equiv 0, \\
G(i, j, x_{-h}, \dots, x_j) &= G_1(i, j, x_{j-\tau}, \dots, x_j) + G_2(i, j, x_{-h}, \dots, x_j), \\
G_1(i, j, 0, \dots, 0) &\equiv G_2(i, j, 0, \dots, 0) \equiv 0.
\end{aligned} \tag{1.7}$$

Here $\tau \ge 0$ is a given integer, and the operator Δ is defined by (1.3).

Step 2 Suppose that the trivial solution of the auxiliary difference equation

$$y_{i+1} = F_1(i, y_{i-\tau}, \dots, y_i) + \sum_{j=0}^{i} G_1(i, j, y_{j-\tau}, \dots, y_j)\xi_{j+1}, \quad i \in Z, \tag{1.8}$$

is asymptotically p-stable and that there exists a Lyapunov functional $v_i = v(i, y_{i-\tau}, \dots, y_i)$ for this equation which satisfies the conditions of Theorem 1.1.

Step 3 The Lyapunov functional V_i is constructed in the form $V_i = V_{1i} + V_{2i}$, where the main component is

$$V_{1i} = v\big(i, x_{i-\tau}, \dots, x_{i-1}, x_i - F_3(i, x_{-h}, \dots, x_i)\big).$$

It is necessary to calculate $\mathbf{E}\Delta V_{1i}$ and in a reasonable way to estimate it.

Step 4 In order to satisfy the conditions of Theorem 1.1 the additional component V_{2i} is usually chosen in some standard way.

Let us make remarks on some peculiarities of this procedure.

(1) It is clear that representation (1.7) in the first step of the procedure is not unique. Hence for different representations one can construct different Lyapunov functionals, and as a result one has different stability conditions.
(2) In the second step for auxiliary equation (1.8) one can choose different Lyapunov functionals v_i. So, for the initial equation (1.1), (1.2) different Lyapunov functionals can be constructed and as a result different stability conditions can be obtained.
(3) It is necessary to stress also that, choosing different ways of estimation of $\mathbf{E}\Delta V_{1i}$ one can construct different Lyapunov functionals and as a result one has different stability conditions.
(4) At last some standard way of the construction of the additional functional V_{2i} sometimes allows one to simplify the fourth step and one then does not use the functional V_{2i} at all. Below, corresponding auxiliary Lyapunov-type theorems are considered.

1.3 Auxiliary Lyapunov-type Theorems

Using the following theorems in some cases it is enough to construct Lyapunov functionals satisfying the conditions which are weaker than (1.5).

Theorem 1.2 *Let there exist a nonnegative functional* $V_{1i} = V_1(i, x_{-h}, \dots, x_i)$ *which satisfies condition* (1.4) *and the conditions*

$$\mathbf{E}\Delta V_{1i} \le a\mathbf{E}|x_i|^p + \sum_{k=-h}^{i} A_{ik}\mathbf{E}|x_k|^p, \quad i \in Z,\ A_{ik} \ge 0, \tag{1.9}$$

$$a + b < 0, \quad b = \sup_{i\in Z} \sum_{j=i}^{\infty} A_{ji}. \tag{1.10}$$

Then the trivial solution of (1.1) *and* (1.2) *is asymptotically* p*-stable.*

Proof Following the procedure of the construction of Lyapunov functionals as described above let us construct the functional V_i in the form $V_i = V_{1i} + V_{2i}$, where V_{1i} satisfies the conditions (1.9), (1.10) and

$$V_{2i} = \sum_{k=-h}^{i-1} |x_k|^p \sum_{j=i}^{\infty} A_{jk}, \quad i = 1, 2, \dots.$$

Calculating $\mathbf{E}\Delta V_{2i}$, we obtain

$$\mathbf{E}\Delta V_{2i} = \mathbf{E}\left(\sum_{k=-h}^{i} |x_k|^p \sum_{j=i+1}^{\infty} A_{jk} - \sum_{k=-h}^{i-1} |x_k|^p \sum_{j=i}^{\infty} A_{jk}\right)$$

$$= \mathbf{E}|x_i|^p \sum_{j=i+1}^{\infty} A_{ji} - \sum_{k=-h}^{i-1} A_{ik}\mathbf{E}|x_k|^p.$$

From this and (1.9) for the functional $V_i = V_{1i} + V_{2i}$ we get $\mathbf{E}\Delta V_i \le (a+b)\mathbf{E}|x_i|^p$. Together with (1.10) this inequality implies (1.5). Thus, there exists a functional V_i that satisfies the conditions of Theorem 1.1, i.e., the trivial solution of (1.1) and (1.2) is asymptotically p-stable. The proof is completed. □

Corollary 1.2 *Let there exist a nonnegative functional $V_i = V(i, x_{-h}, \dots, x_i)$ that satisfies condition* (1.4) *and* (1.9) *with the exact equality instead of the inequality. Then condition* (1.10) *is a necessary and sufficient condition for asymptotic p-stability of the trivial solution of* (1.1).

Proof Sufficiency follows from Theorem 1.2. Necessity is proven similar to Theorem 1.2 and Corollary 1.1. □

Theorem 1.3 *Let there exist a nonnegative functional $V_i = V(i, x_{-h}, \dots, x_i)$ which satisfies conditions* (1.4) *and*

$$\mathbf{E}\Delta V_i \le a\mathbf{E}|x_i|^p + b\mathbf{E}|x_{i-m}|^p, \quad i \in Z,\ m > 0. \tag{1.11}$$

If the solution of (1.1) *and* (1.2) *is p-bounded but is not p-summable then*

$$a + b \ge 0. \tag{1.12}$$

Proof Summing (1.11) from $i = 0$ to $i = N + m$, $N > 0$, we obtain

$$\mathbf{E}V_{N+m+1} - \mathbf{E}V_0 \le a\sum_{i=0}^{N+m} \mathbf{E}|x_i|^p + b\sum_{i=0}^{N+m} \mathbf{E}|x_{i-m}|^p$$

$$= a\sum_{i=0}^{N+m} \mathbf{E}|x_i|^p + b\sum_{i=0}^{N} \mathbf{E}|x_i|^p + b\sum_{i=-m}^{-1} \mathbf{E}|\varphi_i|^p$$

$$= (a+b)\sum_{i=0}^{N} \mathbf{E}|x_i|^p + a\sum_{i=N+1}^{N+m} \mathbf{E}|x_i|^p + b\sum_{i=-m}^{-1} \mathbf{E}|\varphi_i|^p.$$

From this and $V_i \ge 0$ it follows that

$$-(a+b)\sum_{i=0}^{N} \mathbf{E}|x_i|^p \le \mathbf{E}V_0 + a\sum_{i=N+1}^{N+m} \mathbf{E}|x_i|^p + b\sum_{i=-m}^{-1} \mathbf{E}|\varphi_i|^p. \tag{1.13}$$

Since the solution of (1.1) and (1.2) is p-bounded, i.e., $\mathbf{E}|x_i|^p \le C$, $i \in Z$, using (1.4) we have

$$-(a+b)\sum_{i=0}^{N}\mathbf{E}|x_i|^p \le c_1\|\varphi\|^p + m\big(|a|C + |b|\|\varphi\|^p\big) < \infty. \tag{1.14}$$

Let us suppose that (1.12) does not hold, i.e. $a+b<0$. From (1.14) it follows that the solution of (1.1) and (1.2) is p-summable and we obtain a contradiction with the condition of Theorem 1.3. Therefore, (1.12) holds. The proof is completed. □

Corollary 1.3 *Let there exist a nonnegative functional* $V_i = V(i, x_{-h}, \ldots, x_i)$ *which satisfies conditions* (1.4), (1.11) *and*

$$a+b<0. \tag{1.15}$$

Then the solution of (1.1) *and* (1.2) *is* p*-summable or is not* p*-bounded.*

Theorem 1.4 *Let there exist a nonnegative functional* $V_i = V(i, x_{-h}, \ldots, x_i)$ *which satisfies conditions* (1.4), (1.11) *and* (1.15). *If* $a \le 0$ *then the trivial solution of* (1.1) *and* (1.2) *is asymptotically* p*-stable. If* $a>0$, *then each* p*-bounded solution of* (1.1) *and* (1.2) *is asymptotically* p*-trivial.*

Proof From Corollary 1.3 it follows that by conditions (1.4), (1.11), (1.15) each p-bounded solution of (1.1), (1.2) is p-summable and, therefore, it is asymptotically p-trivial. So, it is enough to show that if $a \le 0$, then the trivial solution of (1.1) and (1.2) is p-stable. Note that from (1.11) follows (1.13). If $a=0$, then via (1.13) and (1.4) we have

$$|b|\mathbf{E}|x_N|^p \le |b|\sum_{i=0}^{N}\mathbf{E}|x_i|^p \le \mathbf{E}V_0 \le c_1\|\varphi\|^p,$$

i.e. the trivial solution of (1.1) and (1.2) is p-stable.

Let $a<0$. If $b \le 0$ then condition (1.5) follows from condition (1.11), and from Theorem 1.1 it follows that the trivial solution of (1.1) and (1.2) is asymptotically p-stable. If $b>0$, then condition (1.11) is a particular case of condition (1.9). From this and (1.4), (1.15) it follows that the functional V_i satisfies the conditions of Theorem 1.2 and, therefore, the trivial solution of (1.1), (1.2) is asymptotically p-stable. The proof is completed. □

1.4 Some Useful Lemmas

Lemma 1.1 *Arbitrary positive numbers* $a, b, \alpha, \beta, \gamma$ *satisfy the inequality*

$$a^\alpha b^\beta \le \frac{\alpha}{\alpha+\beta}a^{\alpha+\beta}\gamma^\beta + \frac{\beta}{\alpha+\beta}b^{\alpha+\beta}\gamma^{-\alpha}.$$

Equality is reached for $\gamma = ba^{-1}$.

Proof It is enough to show that the function

$$f(x) = \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta}\gamma^{\beta} + \frac{\beta}{\alpha+\beta} b^{\alpha+\beta}\gamma^{-\alpha} - x^{\alpha}b^{\beta}, \quad x \geq 0,$$

has a minimum in the point $x = b\gamma^{-1}$ and $f(b\gamma^{-1}) = 0$. □

Lemma 1.2 *Arbitrary positive numbers* a, b, γ *and* $\alpha \geq 1$ *satisfy the inequality*

$$(a+b)^{\alpha} \leq (1+\gamma)^{\alpha-1}a^{\alpha} + \left(1+\gamma^{-1}\right)^{\alpha-1}b^{\alpha}.$$

Equality is reached for $\gamma = ba^{-1}$.

Proof For $\alpha = 1$ this inequality is a trivial one. For $\alpha > 1$ it is enough to show that the function

$$f(x) = (1+\gamma)^{\alpha-1}x^{\alpha} + \left(1+\gamma^{-1}\right)^{\alpha-1}b^{\alpha} - (x+b)^{\alpha}, \quad x \geq 0,$$

has a minimum in the point $x = b\gamma^{-1}$ and $f(b\gamma^{-1}) = 0$. □

Lemma 1.3 *Arbitrary vectors* a, b *from* $\mathbf{R}^n$ *and a positive definite* $n \times n$*-matrix* R *satisfy the inequality*

$$a'b + b'a \leq a'Ra + b'R^{-1}b.$$

Proof It is enough to note that

$$a'b + b'a = a'Ra + b'R^{-1}b - (Ra-b)'R^{-1}(Ra-b). \qquad \square$$

Lemma 1.4 *Let* $f(x)$ *be a nonnegative function defined on* $x \in [0, \infty)$. *If* $f(x)$ *is nonincreasing, then*

$$\sum_{l=i+1}^{\infty} f(l) \leq \int_i^{\infty} f(x)\,\mathrm{d}x.$$

If $f(x)$ *is nondecreasing, then*

$$\sum_{l=j}^{i} f(l) \leq \int_j^{i+1} f(x)\,\mathrm{d}x.$$

Consider now conditions for asymptotic mean square stability [123] *of the trivial solution of the linear Ito stochastic differential equation* [87–89]

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + \sigma x(t-h)\dot{w}(t), \tag{1.16}$$

where A, B, σ, $\tau \geq 0$, $h \geq 0$ *are known constants.*

Lemma 1.5 *A necessary and sufficient condition for asymptotic mean square stability of the trivial solution of* (1.16) *is*

$$A + B < 0, \qquad pG < 1, \tag{1.17}$$

where

$$p = \frac{1}{2}\sigma^2, \qquad G = \begin{cases} \frac{Bq^{-1}\sin(q\tau)-1}{A+B\cos(q\tau)}, & B+|A|<0,\ q=\sqrt{B^2-A^2}, \\ \frac{1+|A|\tau}{2|A|}, & B=A<0, \\ \frac{Bq^{-1}\sinh(q\tau)-1}{A+B\cosh(q\tau)}, & A+|B|<0,\ q=\sqrt{A^2-B^2}. \end{cases} \tag{1.18}$$

Proof A necessary and sufficient stability condition (1.17) with

$$G = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}t}{(A+B\cos\tau t)^2 + (t+B\sin\tau t)^2} \tag{1.19}$$

was obtained in [256]. Besides in [208] by virtue of square Lyapunov functional it was shown that the integral (1.19) satisfies the system of the equations

$$GA + \beta(0) + 1 = 0, \qquad GB = \beta(-\tau), \tag{1.20}$$

where the function $\beta(t)$ is a solution of the differential equation

$$\dot\beta(t) = A\beta(t) + B\beta(-t-\tau). \tag{1.21}$$

Suppose that $q^2 = B^2 - A^2 > 0$. Then via (1.21)

$$\begin{aligned} \ddot\beta(t) &= A\dot\beta(t) - B\dot\beta(-t-\tau) \\ &= A\big(A\beta(t) + B\beta(-t-\tau)\big) - B\big(A\beta(-t-\tau) + B\beta(t+\tau-\tau)\big) = -q^2\beta(t) \end{aligned}$$

or

$$\ddot\beta(t) + q^2\beta(t) = 0. \tag{1.22}$$

Substituting the general solution $\beta(t) = C_1\cos qt + C_2\sin qt$ of (1.22) into (1.20) and (1.21) we obtain two equations for G, C_1 and C_2:

$$GA + C_1 = -1, \qquad GB = C_1\cos q\tau - C_2\sin q\tau \tag{1.23}$$

and two homogeneous linear dependent equations for C_1 and C_2:

$$\begin{aligned} C_1(A+B\cos q\tau) - C_2(q+B\sin q\tau) &= 0, \\ C_1(q-B\sin q\tau) + C_2(A-B\cos q\tau) &= 0. \end{aligned} \tag{1.24}$$

Via (1.24)

$$C_2 = C_1\frac{A+B\cos q\tau}{q+B\sin q\tau} = -C_1\frac{q-B\sin q\tau}{A-B\cos q\tau}. \tag{1.25}$$

Substituting the first equality (1.25) into (1.23) and excluding C_1 we obtain

$$G = \frac{A \sin q\tau - q \cos q\tau}{q(A \cos q\tau + q \sin q\tau + B)}. \tag{1.26}$$

Multiplying the numerator and the denominator of the obtained fraction by $B \sin q\tau - q$ one can convert (1.26) to the form of the first line in (1.18). Note that the same result can be obtained using the second equality (1.25).

Suppose now that $q^2 = A^2 - B^2 > 0$. Then similar to (1.22) we obtain the equation $\ddot{\beta}(t) - q^2\beta(t) = 0$ with the general solution $\beta(t) = C_1 e^{qt} + C_2 e^{-qt}$. Substituting this solution into (1.20) and (1.21) similar to (1.23) and (1.24) we have

$$\begin{aligned} &GA + C_1 + C_2 = -1, \qquad GB = C_1 e^{-q\tau} + C_2 e^{q\tau}, \\ &C_1(q - A) - C_2 B e^{q\tau} = 0, \qquad C_1 B + C_2(q + A)e^{q\tau} = 0. \end{aligned} \tag{1.27}$$

Via the two last equations of (1.27)

$$C_2 = C_1 \frac{q - A}{B} e^{-q\tau} = -C_1 \frac{B}{q + A} e^{-q\tau}. \tag{1.28}$$

From the first equality of (1.28) and the two first equations of (1.27) we obtain

$$G = \frac{q - A + Be^{-q\tau}}{q(q - A - Be^{-q\tau})}. \tag{1.29}$$

Put now $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$ (respectively, hyperbolic sine and hyperbolic cosine). Multiplying the numerator and the denominator of (1.29) by $B \sinh q\tau - q$ in the denominator we have

$$\begin{aligned} &\big(q - A - Be^{-q\tau}\big)(B \sinh q\tau - q) \\ &\quad = \big(q - A - Be^{-q\tau}\big)\big(B \cosh q\tau - Be^{-q\tau} - q\big) \\ &\quad = qB \cosh q\tau - AB \cosh q\tau - B^2 e^{-q\tau} \cosh q\tau \\ &\qquad - Bqe^{-q\tau} + ABe^{-q\tau} + B^2 e^{-2q\tau} - A^2 + B^2 + Aq + Bqe^{-q\tau} \\ &\quad = (q - A)(A + B \cosh q\tau) + Be^{-q\tau}\big(A + B\big(e^{q\tau} + e^{-q\tau} - \cosh q\tau\big)\big) \\ &\quad = \big(q - A + Be^{-q\tau}\big)(A + B \cosh q\tau). \end{aligned}$$

As a result we obtain (1.29) in the form of the third line in (1.18). Note also that the same result can be obtained using the second equality of (1.28).

The second line of (1.18) can be obtained from the first (or the third) line as the limit $q \to 0$. The proof is completed. □

Remark 1.2 The necessary and sufficient stability condition in the form (1.17) and (1.18) first was formulated in the Ph.D. thesis of the author (L. Shaikhet. *Some* *problems of stability and optimal control in theory of stochastic systems*. Donetsk, 1981, 120 p., in Russian). The proof of (1.18) in the case $B^2 > A^2$ was taken from [208, 256], the proof of (1.18) in the case $B^2 \le A^2$ was got by the author. Taking into account that the papers [208, 256] are old enough and the proof was not published completely in wide known journals we have decided to put it here.

Chapter 2
Illustrative Example

Here the procedure for the construction of Lyapunov functionals described in Chap. 1 is used for the investigation of p-stability of the scalar difference equation with constant coefficients and $p=2k$, $k=1,2,\dots$ via different representations of the considered equation in the form (1.7) different conditions for asymptotic p-stability are obtained.

2.1 First Way of Construction of Lyapunov Functional

Consider the equation

$$x_{i+1}=ax_i+bx_{i-1}+\sigma x_{i-1}\xi_{i+1}, \quad i\in Z, \tag{2.1}$$

where a, b, σ are known constants, and ξ_i, $i\in Z$, is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i=0$, $\mathbf{E}\xi_i^2=1$.

To begin with, asymptotic mean square stability is investigated, i.e., the case $p=2$.

Using the four steps of the procedure for the construction of Lyapunov functionals, we obtain the following.

1. The right-hand side of (2.1) is already represented in the form (1.7), where $\tau=0$,

$$\begin{aligned}
&F_1(i,x_i)=ax_i, \qquad F_2(i,x_{-1},\dots,x_i)=bx_{i-1},\\
&F_3(i,x_{-1},\dots,x_i)=0, \qquad G_1(i,j,x_j)=0, \quad j=0,\dots,i,\\
&G_2(i,j,x_{-1},\dots,x_j)=0, \quad j=0,\dots,i-1,\\
&G_2(i,i,x_{-1},\dots,x_i)=\sigma x_{i-1}.
\end{aligned}$$

2. The auxiliary equation (1.8) in this case has the form $y_{i+1}=ay_i$. The function $v_i=y_i^2$ is a Lyapunov function for this equation if $|a|<1$, since $\Delta v_i=(a^2-1)y_i^2$.

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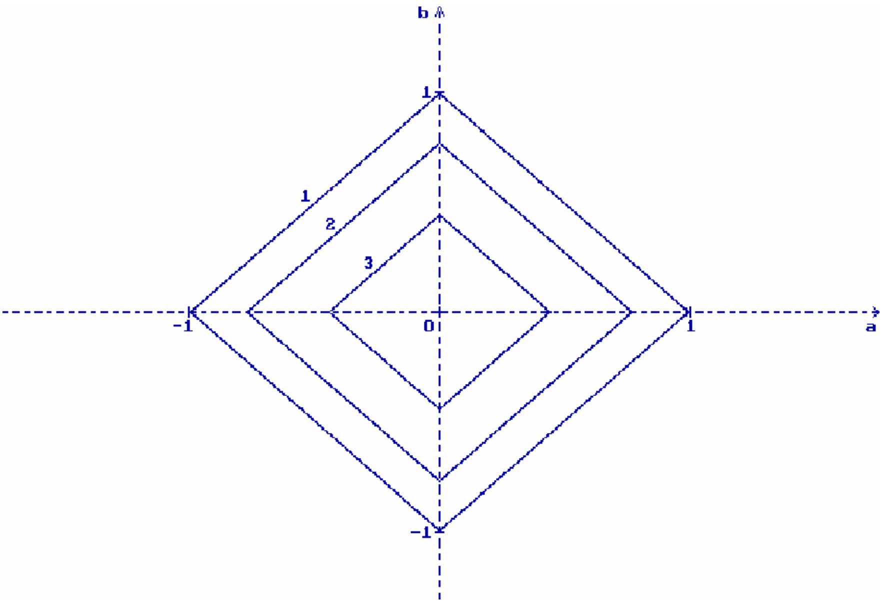


Fig. 2.1 Stability regions for (2.1) given by condition (2.2) for different values of σ^2: (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ has to be chosen in the form $V_{1i} = x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ for (2.1) we get

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = \mathbf{E}\big[(ax_i + bx_{i-1} + \sigma x_{i-1}\xi_{i+1})^2 - x_i^2\big] \\ &\le \big(a^2 - 1 + |ab|\big)\mathbf{E}x_i^2 + A\mathbf{E}x_{i-1}^2, \quad A = b^2 + |ab| + \sigma^2.\end{aligned}$$

4. Choosing the additional functional V_{2i} in the form $V_{2i} = A\mathbf{E}x_{i-1}^2$ we find that the functional $V_i = V_{1i} + V_{2i}$ satisfy the conditions of Theorem 1.1, provided that

$$|a| + |b| < \sqrt{1 - \sigma^2}. \tag{2.2}$$

   So, inequality (2.2) is a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1).

Note, however, that on making the assumption (2.2) the functional V_{1i} satisfies the conditions of Theorem 1.2. So it is an option to use the additional functional V_{2i}.

The stability regions for (2.1) given by inequality (2.2) are shown on Fig. 2.1 for different values of σ^2: (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

2.2 Second Way of Construction of Lyapunov Functional

Let us use now another representation of (2.1).

1. Represent the right-hand side of (2.1) in the form (1.7), where $\tau = 0$,

$$
\begin{aligned}
&F_1(i, x_i) = (a+b)x_i, \qquad F_2(i, x_{-1}, \dots, x_i) = 0,\\
&F_3(i, x_{-1}, \dots, x_i) = -bx_{i-1}, \qquad G_1(i, j, x_j) = 0, \quad j = 0, \dots, i,\\
&G_2(i, j, x_{-1}, \dots, x_j) = 0, \quad j = 0, \dots, i-1,\\
&G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}.
\end{aligned}
$$

2. Auxiliary equation (1.8) in this case is $y_{i+1} = (a+b)y_i$. The function $v_i = y_i^2$ is the Lyapunov function for this equation if $|a+b| < 1$, since $\Delta v_i = ((a+b)^2 - 1)y_i^2$.
3. The functional V_{1i} has be chosen in the form

$$
V_{1i} = (x_i + bx_{i-1})^2. \tag{2.3}
$$

Calculating $\mathbf{E}\Delta V_{1i}$ via (2.1), (2.3), we get

$$
\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[(x_{i+1} + bx_i)^2 - (x_i + bx_{i-1})^2\big]\\
&= \mathbf{E}\big[(a+b-1)x_i + \sigma x_{i-1}\xi_{i+1}\big]\big[(a+b+1)x_i + 2bx_{i-1} + \sigma x_{i-1}\xi_{i+1}\big]\\
&\le \big[(a+b)^2 - 1 + \big|b(a+b-1)\big|\big]\mathbf{E}x_i^2 + B\mathbf{E}x_{i-1}^2,\\
B &= \sigma^2 + \big|b(a+b-1)\big|.
\end{aligned}
$$

Via Theorem 1.2 we find that the inequality

$$
(a+b)^2 + 2\big|b(a+b-1)\big| + \sigma^2 < 1 \tag{2.4}
$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1).

Note that condition (2.4) can also be written in the form

$$
|a+b| < \sqrt{1-\sigma^2}, \quad \sigma^2 < (1-a-b)\big(1+a+b-2|b|\big). \tag{2.5}
$$

The stability regions defined by conditions (2.5) are shown on Fig. 2.2 for different values of σ^2: (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

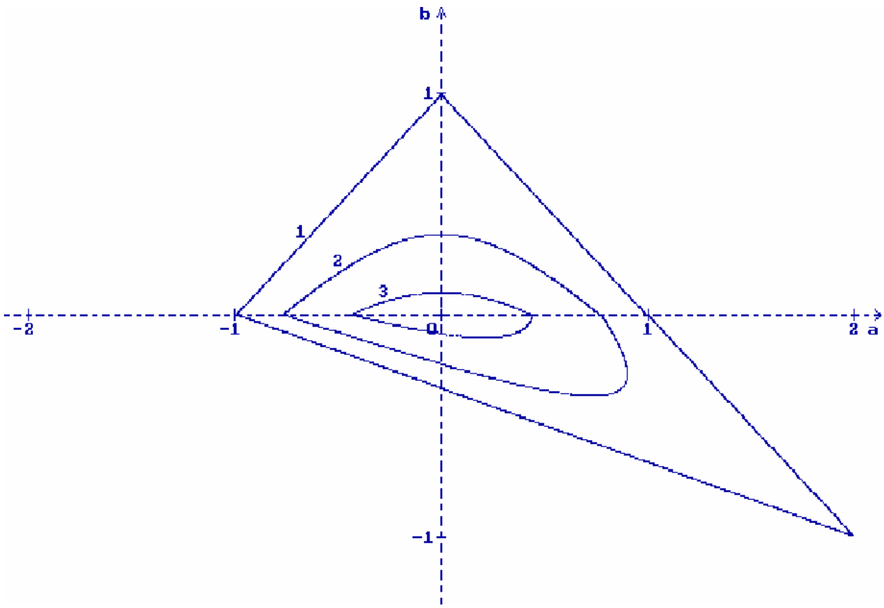


Fig. 2.2 Stability regions for (2.1) given by condition (2.5) for different values of σ^2: (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

2.3 Third Way of Construction of Lyapunov Functional

In some cases new stability conditions can be obtained by iterating the right-hand side of the equation under consideration. For example, from (2.1) we have

$$\begin{aligned} x_{i+1} &= a(ax_{i-1} + bx_{i-2} + \sigma x_{i-2}\xi_i) + bx_{i-1} + \sigma x_{i-1}\xi_{i+1} \\ &= \big(a^2 + b\big)x_{i-1} + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}. \end{aligned} \tag{2.6}$$

1. Representing the right-hand side of (2.6) in the form (1.7), put $\tau = 0$,

$$F_1(i, x_i) = F_3(i, x_{-1}, \dots, x_i) = 0,$$

$$F_2(i, x_{-1}, \dots, x_i) = \big(a^2 + b\big)x_{i-1} + abx_{i-2},$$

$$G_1(i, j, x_j) = 0, \qquad j = 0, \dots, i,$$

$$G_2(i, j, x_{-1}, \dots, x_j) = 0, \qquad j = 0, \dots, i - 2,$$

$$G_2(i, i - 1, x_{-1}, \dots, x_{i-1}) = a\sigma x_{i-2}, \qquad G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}.$$

2. The auxiliary equation is $y_{i+1} = 0$, $i \in Z$. The function $v_i = y_i^2$ is the Lyapunov function for this equation, since $\Delta v_i = y_{i+1}^2 - y_i^2 = -y_i^2$.

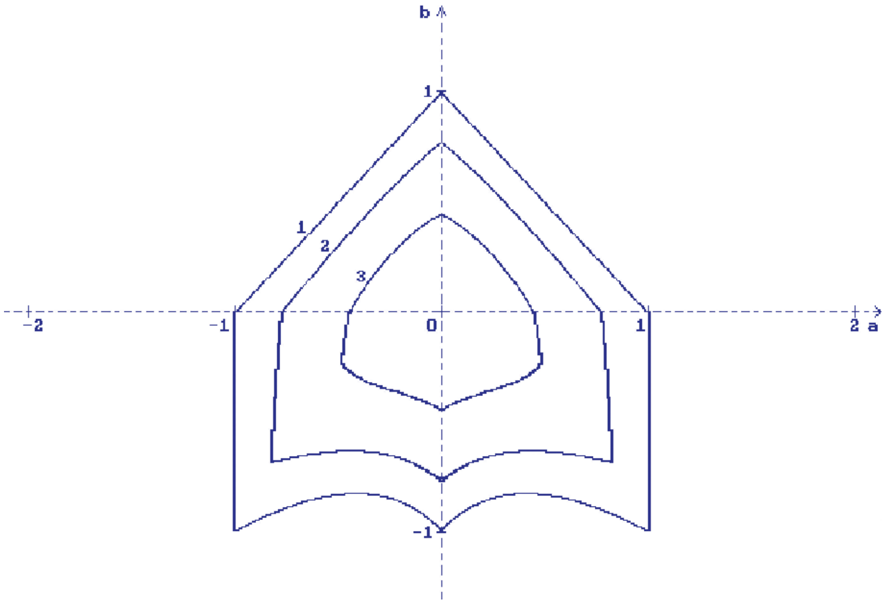


Fig. 2.3 Stability regions for (2.1) given by condition (2.7) for different values of σ^2: (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

3. The functional V_{1i} has to be chosen in the form $V_{1i} = x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ via (2.6), we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big)\\ &= \mathbf{E}\big[\big(\big(a^2+b\big)x_{i-1} + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}\big)^2 - x_i^2\big]\\ &\le -\mathbf{E}x_i^2 + A_1\mathbf{E}x_{i-1}^2 + A_2\mathbf{E}x_{i-2}^2,\\ A_1 &= |ab|\big|a^2+b\big| + \big(a^2+b\big)^2 + \sigma^2,\\ A_2 &= |ab|\big|a^2+b\big| + a^2b^2 + \sigma^2a^2.\end{aligned}$$

It follows from Theorem 1.2 that a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) has the form $A_1 + A_2 < 1$ or

$$\big|a^2+b\big| + |ab| < \sqrt{1 - \sigma^2\big(1+a^2\big)}. \tag{2.7}$$

Stability regions defined by condition (2.7) are shown on Fig. 2.3 for different values of σ^2: (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

Using one more iteration, via (2.6) and (2.1) we obtain

$$\begin{aligned}x_{i+1} &= \big(a^2+b\big)(ax_{i-2} + bx_{i-3} + \sigma x_{i-3}\xi_{i-1})\\ &\quad + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}\end{aligned}$$

$$= a(a^2+2b)x_{i-2} + b(a^2+b)x_{i-3}$$
$$+ \sigma(a^2+b)x_{i-3}\xi_{i-1} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}.$$

For $V_{1i} = x_i^2$ we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}(x_{i+1}^2 - x_i^2) \\
&= \big[\big(a(a^2+2b)x_{i-2} + b(a^2+b)x_{i-3} \\
&\quad + \sigma(a^2+b)x_{i-3}\xi_{i-1} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}\big)^2 - x_i^2\big] \\
&\le -\mathbf{E}x_i^2 + \sigma^2\mathbf{E}x_{i-1}^2 + B_2\mathbf{E}x_{i-2}^2 + B_3\mathbf{E}x_{i-3}^2, \\
B_2 &= a^2(a^2+2b)^2 + \big|ab(a^2+b)(a^2+2b)\big| + \sigma^2 a^2, \\
B_3 &= b^2(a^2+b)^2 + \big|ab(a^2+b)(a^2+2b)\big| + \sigma^2(a^2+b)^2.
\end{aligned}$$

Via Theorem 1.2 a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) takes the form $\sigma^2 + B_2 + B_3 < 1$ or

$$\big|a(a^2+2b)\big| + \big|b(a^2+b)\big| < \sqrt{1-\sigma^2\big(1+a^2+(a^2+b)^2\big)}. \tag{2.8}$$

After the next iteration one can obtain a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) in the form

$$\begin{aligned}
&\big|a^2(a^2+2b) + b(a^2+b)\big| + \big|ab(a^2+2b)\big| \\
&\quad < \sqrt{1-\sigma^2\big(1+a^2+(a^2+b)^2+a^2(a^2+2b)^2\big)}.
\end{aligned} \tag{2.9}$$

Stability regions defined by conditions (2.7) (the bound number 1), (2.8) (the bound number 2) and (2.9) (the bound number 3) are shown on Fig. 2.4 for $\sigma^2 = 0$. One can see that each next iteration increases the stability region.

2.4 Fourth Way of Construction of Lyapunov Functional

Consider now the case $\tau = 1$.

1. Represent (2.1) in the form (1.7) using

$$\begin{aligned}
&F_1(i, x_{i-1}, x_i) = ax_i + bx_{i-1}, \qquad G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}, \\
&F_2(i, x_{-1}, \dots, x_i) = F_3(i, x_{-1}, \dots, x_i) = 0, \\
&G_1(i, j, x_j) = 0, \quad j = 0, \dots, i, \\
&G_2(i, j, x_{-1}, \dots, x_j) = 0, \quad j = 0, \dots, i-1.
\end{aligned}$$

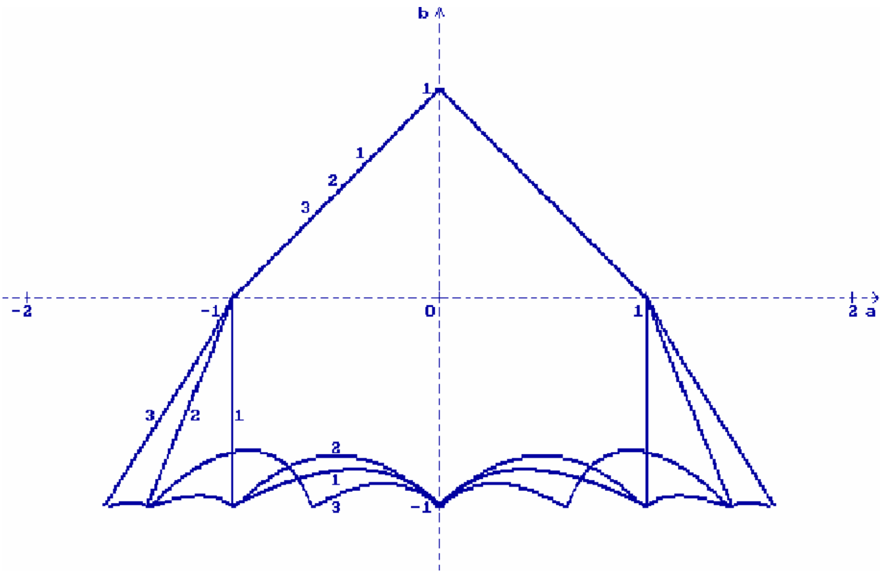


Fig. 2.4 Stability regions for (2.1) given for $\sigma^2 = 0$ by the conditions: (*1*) (2.7), (*2*) (2.8), (*3*) (2.9)

Put

$$x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}.$$

Then (2.1) can be written in the matrix form

$$x(i+1) = Ax(i) + Bx(i)\xi_{i+1}. \tag{2.10}$$

2. Introduce the vector $y(i)$ such that $y'(i) = (y_{i-1}, y_i)$. Then the auxiliary equation has the form

$$y(i+1) = Ay(i). \tag{2.11}$$

Let C be an arbitrary positive semidefinite symmetric matrix. Suppose that the solution D of the matrix equation

$$A'DA - D = -C \tag{2.12}$$

is a positive semidefinite matrix with $d_{22} > 0$. In this case the function $v_i = y'(i)Dy(i)$ is the Lyapunov function for (2.11). In particular, if

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad c_1 \geq 0,\ c_2 > 0, \tag{2.13}$$

then the elements of the matrix D are

$$d_{11} = c_1 + b^2 d_{22}, \qquad d_{12} = \frac{ab}{1-b} d_{22},$$
$$d_{22} = \frac{(c_1 + c_2)(1-b)}{(1+b)[(1-b)^2 - a^2]}. \tag{2.14}$$

The matrix D is positive semidefinite with $d_{22} > 0$ if and only if

$$|b| < 1, \qquad |a| < 1 - b. \tag{2.15}$$

3. The functional V_{1i} has be chosen in the form $V_{1i} = x'(i)Dx(i)$. Calculating $\mathbf{E}\Delta V_{1i}$, via (2.10), (2.11) and (2.13) we get

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big] \\ &= \mathbf{E}\big[\big(Ax(i) + Bx(i)\xi_{i+1}\big)' D\big(Ax(i) + Bx(i)\xi_{i+1}\big) - x'(i)Dx(i)\big] \\ &= -\mathbf{E}x'(i)Cx(i) + \sigma^2 d_{22}\mathbf{E}x_{i-1}^2 \\ &= -c_2\mathbf{E}x_i^2 + \big(\sigma^2 d_{22} - c_1\big)\mathbf{E}x_{i-1}^2.\end{aligned}$$

Let us suppose that

$$\frac{\sigma^2(1-b)}{(1+b)[(1-b)^2 - a^2]} < 1. \tag{2.16}$$

Then $\sigma^2 d_{22} < c_1 + c_2$. Choosing $c_1 = \sigma^2 d_{22}$ via Corollary 1.2 we find that the inequalities (2.15), (2.16) are necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of (2.1).

In particular, if $a = b = 0$, then the inequality $\sigma^2 < 1$ is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the equation $x_{i+1} = \sigma x_{i-1}\xi_{i+1}$. In fact, in this case we have $\mathbf{E}x_{2i-k}^2 = \sigma^{2i}\mathbf{E}x_{-k}^2$, $k = 0, 1$; $i = 0, 1, \dots$.

Remark 2.1 It is easy to see that the stability conditions (2.15) and (2.16) do not depend on c_1, c_2. So, for simplicity one can put $c_1 = 0$, $c_2 = 1$.

Stability regions defined by conditions (2.15), (2.16) are shown on Fig. 2.5 for different values of σ^2: (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

On Fig. 2.6 one can see the stability regions corresponding to conditions (2.2), (2.5), (2.7) and (2.15), (2.16) for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.8$.

2.5 One Generalization

Let us obtain a condition for asymptotic p-stability of the trivial solution of (2.1) in the case $p = 2k$, $k = 1, 2, \dots$.

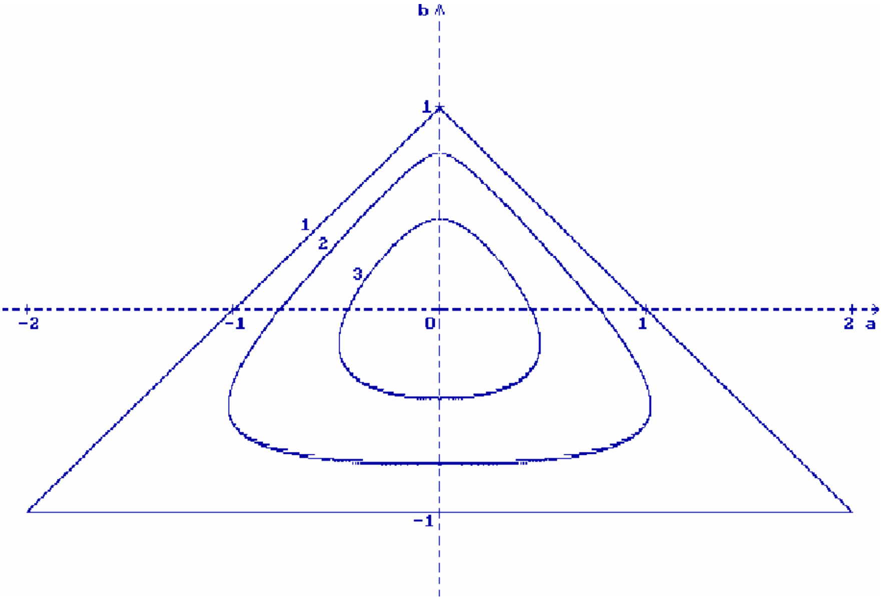


Fig. 2.5 Stability regions for (2.1) given by the conditions (2.15) and (2.16), for different values of σ^2: (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

Put $\lambda_r = \mathbf{E}\xi_i^r$, $r = 1, \ldots, 2k$, and suppose that ξ_i is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\lambda_{2m-1} = 0$, $\lambda_{2m} < \infty$, $m = 1, \ldots, k$.

Choose $V_{1i} = x_i^{2k}$. Then for (2.1) we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[(ax_i + bx_{i-1} + \sigma x_{i-1}\xi_{i+1})^{2k} - x_i^{2k}\big] \\ &= \mathbf{E}\left[\sum_{j=0}^{2k} C_{2k}^j (ax_i + bx_{i-1})^j (\sigma x_{i-1}\xi_{i+1})^{2k-j} - x_i^{2k}\right].\end{aligned}$$

Using $\lambda_{2m-1} = 0$, $\lambda_0 = 1$ and putting $j = 2l$, we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \sum_{l=0}^{k} C_{2k}^{2l}\mathbf{E}(ax_i + bx_{i-1})^{2l}(\sigma x_{i-1})^{2(k-l)}\lambda_{2(k-l)} - \mathbf{E}x_i^{2k} \\ &= \sum_{l=0}^{k} C_{2k}^{2l}\mathbf{E}\left(\sum_{j=0}^{2l} C_{2l}^j (ax_i)^j (bx_{i-1})^{2l-j}\right)(\sigma x_{i-1})^{2(k-l)}\lambda_{2(k-l)} - \mathbf{E}x_i^{2k} \\ &\le \sum_{l=0}^{k} C_{2k}^{2l}\sum_{j=0}^{2l} C_{2l}^j |a|^j |b|^{2l-j}\sigma^{2(k-l)}\lambda_{2(k-l)}\mathbf{E}|x_i|^j |x_{i-1}|^{2k-j} - \mathbf{E}|x_i|^{2k}.\end{aligned}$$

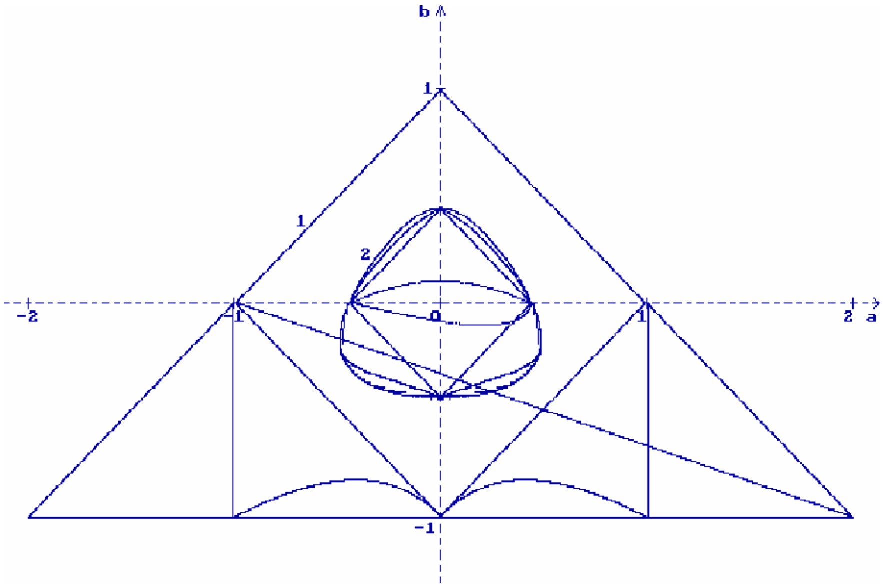


Fig. 2.6 Stability regions for (2.1) given by conditions (2.2), (2.5) and (2.7), and (2.15) and (2.16) for two values of σ^2: (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.8$

Via Lemma 1.1

$$|x_i|^j |x_{i-1}|^{2k-j} \le \frac{j}{2k}|x_i|^{2k} + \frac{2k-j}{2k}|x_{i-1}|^{2k}.$$

So,

$$\mathbf{E}\Delta V_{1i} \le \sum_{l=0}^{k} C_{2k}^{2l} \sum_{j=0}^{2l} C_{2l}^{j} |a|^j |b|^{2l-j} \sigma^{2(k-l)} \lambda_{2(k-l)} \times \left(\frac{j}{2k}\mathbf{E}|x_i|^{2k} \frac{2k-j}{2k}\mathbf{E}|x_{i-1}|^{2k}\right) - \mathbf{E}|x_i|^{2k}.$$

Using Theorem 1.2 we obtain the following sufficient condition for asymptotic $2k$-stability of the trivial solution of (2.1):

$$\sum_{l=0}^{k} C_{2k}^{2l} \big(|a| + |b|\big)^{2l} \sigma^{2(k-l)} \lambda_{2(k-l)} < 1. \tag{2.17}$$

Note that for $k = 1$ and $\lambda_2 = 1$ condition (2.17) coincides with (2.2). If $k = 2$, then condition (2.17) gives

$$\big(|a| + |b|\big)^4 + 6\lambda_2\sigma^2\big(|a| + |b|\big)^2 + \lambda_4\sigma^4 < 1$$

or

$$|a| + |b| < \sqrt{\sqrt{1 + \left(9\lambda_2^2 - \lambda_4\right)\sigma^4} - 3\lambda_2\sigma^2}, \quad \lambda_4\sigma^4 < 1. \tag{2.18}$$

In particular, if ξ_i is Gaussian, then $\lambda_4 = 3\lambda_2^2$ and the stability condition (2.18) takes the form

$$|a| + |b| < \sqrt{\sqrt{1 + 6\lambda_2^2\sigma^4} - 3\lambda_2\sigma^2}, \quad 3\lambda_2^2\sigma^4 < 1.$$

Chapter 3
Linear Equations with Stationary Coefficients

Here the procedure of the construction of Lyapunov functionals described in Chap. 1 is applied to the equation with stationary coefficients. Four different ways of the construction of Lyapunov functionals are shown.

3.1 First Way of the Construction of the Lyapunov Functional

Consider the equation

$$\begin{aligned} x_{i+1} &= \sum_{l=-h}^{i} a_{i-l} x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_{j+1}, \quad i \in Z, \\ x_i &= \varphi_i, \quad i \in Z_0, \end{aligned} \tag{3.1}$$

where a_i and σ_j^i are known constants, ξ_i is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

Below, the following notations are used:

$$\begin{aligned} &k_m = \max(k, 0), \qquad \alpha = \sum_{l=1}^{\infty} \left| \sum_{m=l}^{\infty} a_m \right|, \qquad \alpha_1 = \sum_{l=0}^{\infty} |a_l|, \\ &S_0 = \sum_{p=0}^{\infty} \left(\sum_{l=0}^{\infty} |\sigma_l^p| \right)^2, \qquad S_r = \sum_{i=r}^{\infty} \sum_{j=0}^{\infty} |\sigma_j^i|, \quad r = 1, 2, \dots, \\ &\eta_i = \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_{j+1}, \quad i = 0, 1, 2, \dots. \end{aligned} \tag{3.2}$$

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_3,

1. Represent the right-hand side of (3.1) in the form (1.7) with $\tau = 0$,

$$F_1(i, x_i) = a_0 x_i, \qquad F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-1} a_{i-l} x_l,$$

$$F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i, j, x_j) = 0, \qquad G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l,$$

$$j = 0, \dots, i, \; i = 0, 1, \dots.$$

2. The auxiliary difference equation (1.8) in this case is $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0| < 1$, since $\Delta v_i = (a_0^2 - 1) y_i^2$.
3. The functional V_{1i} has to be chosen in the form $V_{1i} = x_i^2$.
4. Calculating $\mathbf{E}\Delta V_{1i}$, via (3.1) and (3.2) we have

$$\mathbf{E}\Delta V_{1i} = \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = \mathbf{E}\left(\sum_{l=-h}^{i} a_{i-l} x_l + \eta_i\right)^2 - \mathbf{E}x_i^2 = -\mathbf{E}x_i^2 + \sum_{k=1}^{3} I_k,$$

where

$$I_1 = \mathbf{E}\left(\sum_{l=-h}^{i} a_{i-l} x_l\right)^2, \qquad I_2 = 2\mathbf{E}\eta_i \sum_{l=-h}^{i} a_{i-l} x_l, \qquad I_3 = \mathbf{E}\eta_i^2.$$

Via (3.2)

$$I_1 \le \alpha_1 \sum_{l=-h}^{i} |a_{i-l}| \mathbf{E}x_l^2.$$

Since $\mathbf{E}x_k x_l \xi_{j+1} = 0$ for $k, l \le j$ and $\mathbf{E}(x_k \xi_{j+1})^2 = \mathbf{E}x_k^2$ for $k \le j$ then

$$\begin{aligned}
|I_2| &= 2\left|\mathbf{E}\eta_i \left(\sum_{l=-h}^{j} a_{i-l} x_l + \sum_{l=j+1}^{i} a_{i-l} x_l\right)\right| \\
&= 2\left|\mathbf{E}\sum_{j=0}^{i-1} \sum_{k=-h}^{j} \sigma_{j-k}^{i-j} x_k \xi_{j+1} \sum_{l=j+1}^{i} a_{i-l} x_l\right| \\
&\le \sum_{j=0}^{i-1} \sum_{k=-h}^{j} \sum_{l=j+1}^{i} \big|\sigma_{j-k}^{i-j}\big| |a_{i-l}| \big(\mathbf{E}x_l^2 + \mathbf{E}x_k^2\big)
\end{aligned}$$

$$= \sum_{l=1}^{i}\left(|a_{i-l}|\sum_{j=0}^{l-1}\sum_{k=-h}^{j}\left|\sigma_{j-k}^{i-j}\right|\right)\mathbf{E}x_l^2 + \sum_{k=-h}^{i-1}\left(\sum_{j=k_m}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=j+1}^{i}|a_{i-l}|\right)\mathbf{E}x_k^2.$$

Note that via (3.2) for $l \le i$

$$\sum_{j=0}^{l-1}\sum_{k=-h}^{j}\left|\sigma_{j-k}^{i-j}\right| \le \sum_{j=0}^{i-1}\sum_{k=-h}^{j}\left|\sigma_{j-k}^{i-j}\right| = \sum_{j=1}^{i}\sum_{k=0}^{i-1+h}\left|\sigma_k^j\right| \le S_1$$

and

$$\sum_{j=k_m}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=j+1}^{i}|a_{i-l}| = \sum_{j=k_m}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=0}^{i-j-1}|a_l| \le \alpha_1\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|.$$

So,

$$|I_2| \le S_1\sum_{k=1}^{i}|a_{i-k}|\mathbf{E}x_k^2 + \alpha_1\sum_{k=-h}^{i-1}\sum_{p=1}^{k_m}\left|\sigma_{i-k-p}^{p}\right|\mathbf{E}x_k^2.$$

Since

$$\begin{aligned}
I_3 &= \sum_{j=0}^{i}\mathbf{E}\left(\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\right)^2 \le \sum_{j=0}^{i}\left(\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\right)\left(\sum_{k=-h}^{j}\left|\sigma_{j-k}^{i-j}\right|\mathbf{E}x_k^2\right)\\
&= \sum_{p=0}^{i}\left(\sum_{l=-h}^{i-p}\left|\sigma_{i-p-l}^{p}\right|\right)\left(\sum_{k=-h}^{i-p}\left|\sigma_{i-p-k}^{p}\right|\mathbf{E}x_k^2\right)\\
&\le \sum_{p=0}^{i}\left(\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)\left(\sum_{k=-h}^{i-p}\left|\sigma_{i-p-k}^{p}\right|\mathbf{E}x_k^2\right)\\
&= \sum_{k=-h}^{i}\left(\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)\mathbf{E}x_k^2,
\end{aligned}$$

then

$$\mathbf{E}\Delta V_{1i} \le -\mathbf{E}x_i^2 + \sum_{k=-h}^{i}A_{ik}\mathbf{E}x_k^2,$$

where

$$A_{ik} = (\alpha_1 + S_1)|a_{i-k}| + \alpha_1\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right| + \sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|.$$

Note that $A_{j+i,i}$ for $i \in Z$ depends on j only. So,

$$\begin{aligned}\sum_{j=0}^{\infty} A_{j+i,i} &= \sum_{j=0}^{\infty}\left((\alpha_1 + S_1)|a_j| + \alpha_1 \sum_{p=1}^{j}|\sigma_{j-p}^{p}| + \sum_{p=0}^{j}\sigma_{j-p}^{p}\sum_{l=0}^{\infty}|\sigma_l^p|\right) \\ &\le (\alpha_1 + S_1)\alpha_1 + \alpha_1 \sum_{j=0}^{\infty}\sum_{p=1}^{j}|\sigma_{j-p}^{p}| + \sum_{j=0}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^{p}|\sum_{l=0}^{\infty}|\sigma_l^p| \\ &\le \alpha_1^2 + 2\alpha_1 S_1 + S_0.\end{aligned}$$

Via Theorem 1.2 we find that the inequality

$$\alpha_1^2 + 2\alpha_1 S_1 + S_0 < 1 \tag{3.3}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (3.1).

In particular, for (2.1) we have $\alpha_1 = |a_0| + |a_1|$, $S_0 = \sigma^2$, $S_1 = 0$, and from (3.3) condition (2.2) follows.

3.2 Second Way of the Construction of the Lyapunov Functional

1. Represent the right-hand side of (3.1) in the form (1.7) with

$$\begin{aligned}&\tau = 0, \qquad F_1(i, x_i) = \beta x_i, \quad \beta = \sum_{j=0}^{\infty} a_j, \\ &F_2(i, x_{-h}, \dots, x_i) = 0, \\ &F_3(i) = F_3(i, x_{-h}, \dots, x_i) = -\sum_{l=-h}^{i-1} x_l \sum_{j=i-l}^{\infty} a_j, \\ &G_1(i, j, x_j) = 0, \qquad G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l, \\ &j = 0, \dots, i, \quad i = 0, 1, \dots.\end{aligned} \tag{3.4}$$

2. Auxiliary equation (1.8) in this case is $y_{i+1} = \beta y_i$. Below it is supposed that $|\beta| < 1$. By this condition the function $v_i = y_i^2$ is a Lyapunov function for the auxiliary equation, since $\Delta v_i = (\beta^2 - 1) y_i^2$.

3. The functional V_{1i} has to be chosen in the form $V_{1i}=(x_i-F_3(i))^2$.
4. Via the obtained representation $x_{i+1}=\beta x_i+\Delta F_3(i)+\eta_i$, we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i}&=\mathbf{E}\big[\big(x_{i+1}-F_3(i+1)\big)^2-\big(x_i-F_3(i)\big)^2\big]\\&=\mathbf{E}\big(x_{i+1}-F_3(i+1)-x_i+F_3(i)\big)\big(x_{i+1}-F_3(i+1)+x_i-F_3(i)\big)\\&=\mathbf{E}\big(\beta x_i+\Delta F_3(i)+\eta_i-F_3(i+1)-x_i+F_3(i)\big)\\&\quad\times\big(\beta x_i+\Delta F_3(i)+\eta_i-F_3(i+1)+x_i-F_3(i)\big)\\&=\mathbf{E}\big((\beta-1)x_i+\eta_i\big)\big((\beta+1)x_i+\eta_i-2F_3(i)\big)=\sum_{k=1}^{5}I_k,\end{aligned}$$

where

$$I_1=\big(\beta^2-1\big)\mathbf{E}x_i^2,\qquad I_2=2\beta\mathbf{E}x_i\eta_i,\qquad I_3=\mathbf{E}\eta_i^2,$$
$$I_4=2(1-\beta)\mathbf{E}x_iF_3(i),\qquad I_5=-2\mathbf{E}\eta_iF_3(i).$$

Let us estimate the summands $I_2,\dots,I_5$ using the representations for $F_3(i)$, η_i and properties of random variables ξ_j. Since $\mathbf{E}x_i\sum_{l=-h}^{i}\sigma_{i-l}^{0}x_l\xi_{i+1}=0$, for I_2 we have

$$|I_2|\le|\beta|\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\big|\sigma_{j-l}^{i-j}\big|\big(\mathbf{E}x_i^2+\mathbf{E}x_l^2\big)\le|\beta|S_1\mathbf{E}x_i^2+|\beta|\sum_{k=-h}^{i-1}\sum_{p=1}^{i-k_m}\big|\sigma_{i-k-p}^{p}\big|\mathbf{E}x_k^2.$$

For I_3:

$$\begin{aligned}I_3&=\sum_{j=0}^{i}\mathbf{E}\left(\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\right)^2\le\sum_{j=0}^{i}\sum_{l=-h}^{j}\big|\sigma_{j-l}^{i-j}\big|\sum_{k=-h}^{j}\big|\sigma_{j-k}^{i-j}\big|\mathbf{E}x_k^2\\&\le\big|\sigma_0^0\big|\sum_{l=0}^{\infty}\big|\sigma_l^0\big|\mathbf{E}x_i^2+\sum_{k=-h}^{i-1}\sum_{p=0}^{i-k_m}\big|\sigma_{i-k-p}^{p}\big|\sum_{l=0}^{\infty}\big|\sigma_l^p\big|\mathbf{E}x_k^2.\end{aligned}$$

Via (3.2) and $|\beta|<1$

$$\begin{aligned}|I_4|&\le(1-\beta)\sum_{l=-h}^{i-1}\left|\sum_{j=i-l}^{\infty}a_j\right|\big(\mathbf{E}x_i^2+\mathbf{E}x_l^2\big)\\&\le(1-\beta)\left(\alpha\mathbf{E}x_i^2+\sum_{k=-h}^{i-1}\left|\sum_{j=i-k}^{\infty}a_j\right|\mathbf{E}x_k^2\right).\end{aligned}$$

Since $\mathbf{E}x_lx_k\xi_j=0$ for $k\le j$, $l\le j$, then via (3.2) we obtain

$$
\begin{aligned}
|I_5| &= 2\left|\mathbf{E}\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\left(\sum_{k=-h}^{j}x_k\sum_{m=i-k}^{\infty}a_m+\sum_{k=j+1}^{i-1}x_k\sum_{m=i-k}^{\infty}a_m\right)\right| \\
&= 2\left|\mathbf{E}\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\sum_{k=j+1}^{i-1}x_k\sum_{m=i-k}^{\infty}a_m\right| \\
&\le \sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_m\right|\left(\mathbf{E}x_l^2+\mathbf{E}x_k^2\right) \\
&= \sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_m\right|\mathbf{E}x_l^2+\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_m\right|\mathbf{E}x_k^2 \\
&= \sum_{k=-h}^{i-2}\left(\sum_{j=k_m}^{i-2}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=j+1}^{i-1}\left|\sum_{m=i-l}^{\infty}a_m\right|\right)\mathbf{E}x_k^2 \\
&\quad+\sum_{k=1}^{i-1}\left(\left|\sum_{m=i-k}^{\infty}a_m\right|\sum_{j=0}^{k-1}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\right)\mathbf{E}x_k^2 \\
&\le \alpha\sum_{k=-h}^{i-2}\sum_{p=2}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\mathbf{E}x_k^2+S_2\sum_{k=1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_m\right|\mathbf{E}x_k^2.
\end{aligned}
$$

Thus, as a result we obtain

$$
\mathbf{E}\Delta V_{1i}\le\left(\beta^2-1+|\beta|S_1+\left|\sigma_0^0\right|\sum_{l=0}^{\infty}\left|\sigma_l^0\right|+(1-\beta)\alpha\right)\mathbf{E}x_i^2+\sum_{k=-h}^{i-1}B_{ik}\mathbf{E}x_k^2,
$$

where

$$
\begin{aligned}
B_{ik} &= |\beta|\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|+\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right| \\
&\quad+(1-\beta)\left|\sum_{j=i-k}^{\infty}a_j\right|+\alpha\sum_{p=2}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|+S_2\left|\sum_{m=i-k}^{\infty}a_m\right|.
\end{aligned}
$$

It is easy to see that $B_{j+i,i}$ for $i\ge 0$ depends on j only. So,

$$
\begin{aligned}
\sum_{j=1}^{\infty}B_{j+i,i} &= |\beta|\sum_{j=1}^{\infty}\sum_{p=1}^{j}\left|\sigma_{j-p}^{p}\right|+\sum_{j=1}^{\infty}\sum_{p=0}^{j}\left|\sigma_{j-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right| \\
&\quad+(1-\beta)\sum_{j=1}^{\infty}\left|\sum_{m=j}^{\infty}a_m\right|+\alpha\sum_{j=2}^{\infty}\sum_{p=2}^{j}\left|\sigma_{j-p}^{p}\right|+S_2\sum_{j=1}^{\infty}\left|\sum_{m=j}^{\infty}a_m\right| \\
&= |\beta|S_1+(1-\beta)\alpha+2\alpha S_2+\sum_{j=1}^{\infty}\sum_{p=0}^{j}\left|\sigma_{j-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|.
\end{aligned}
$$

Via Theorem 1.2 we obtain the following sufficient condition for asymptotic mean square stability of the trivial solution of (3.1):

$$|\beta| < 1, \quad \beta^2 + 2\alpha(1-\beta) + S_0 + 2\big(|\beta|S_1 + \alpha S_2\big) < 1 \tag{3.5}$$

or otherwise

$$|\beta| < 1, \quad S_0 + 2\big(|\beta|S_1 + \alpha S_2\big) < (1-\beta)(1+\beta-2\alpha). \tag{3.6}$$

In particular, for (2.1) we have $\beta = a_0 + a_1$, $\alpha = |a_1|$, $S_0 = \sigma^2$, $S_1 = S_2 = 0$, and from (3.6) condition (2.5) follows.

3.3 Third Way of the Construction of the Lyapunov Functional

1. Represent the right-hand side of (3.1) in the form (1.7) with

$$\tau = 1, \qquad F_1(i, x_{i-1}, x_i) = a_0 x_i + a_1 x_{i-1},$$

$$F_2(i) = F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-2} a_{i-l} x_l, \qquad F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i, j, x_{j-1}, x_j) = 0, \qquad G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l,$$

$$j = 0, \dots, i, \; i = 0, 1, \dots.$$

   Put

$$x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ a_1 & a_0 \end{pmatrix}, \qquad B(i) = \begin{pmatrix} 0 \\ b(i) \end{pmatrix}, \qquad b(i) = F_2(i) + \eta_i. \tag{3.7}$$

   Then (3.1) can be written in a matrix form

$$x(i+1) = Ax(i) + B(i). \tag{3.8}$$

2. Introduce the vector $y(i) = (y_{i-1}, y_i)'$. The auxiliary equation is

$$y(i+1) = Ay(i). \tag{3.9}$$

   Using Remark 2.1, consider the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.10}$$

The solution of (3.9) is a symmetric matrix D with the elements d_{ij}, such that

$$d_{11} = a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1-a_1} d_{22},$$
$$d_{22} = \frac{1-a_1}{(1+a_1)[(1-a_1)^2 - a_0^2]}. \tag{3.11}$$

By the conditions

$$|a_1| < 1, \qquad |a_0| < 1 - a_1, \tag{3.12}$$

we obtain $d_{22} > 0$ and the function $v_i = y'(i)Dy(i)$ is a Lyapunov function for the auxiliary equation (3.8).

3. The functional V_{1i} has to be chosen in the form $V_{1i} = x'(i)Dx(i)$.
4. Calculating $\mathbf{E}\Delta V_{1i}$, via (3.7) and (3.10) we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big] \\
&= \mathbf{E}\big[\big(Ax(i) + B(i)\big)' D\big(Ax(i) + B(i)\big) - x'(i)Dx(i)\big] \\
&= -\mathbf{E}x_i^2 + d_{22}\mathbf{E}b^2(i) + 2\mathbf{E}\big(d_{22}a_1 x_{i-1} + (d_{12} + d_{22}a_0)x_i\big)b(i).
\end{aligned}$$

Using (3.11) we obtain $d_{12} + d_{22}a_0 = d_{22}a_0(1-a_1)^{-1}$. From this and (3.7) it follows that

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= -\mathbf{E}x_i^2 + d_{22}\mathbf{E}b^2(i) + 2d_{22}\mathbf{E}\left(a_1 x_{i-1} + \frac{a_0}{1-a_1} x_i\right) b(i) \\
&= -\mathbf{E}x_i^2 + \sum_{j=1}^{7} I_j,
\end{aligned}$$

where

$$I_1 = d_{22}\mathbf{E}F_2^2(i), \qquad I_2 = 2d_{22}\eta_i F_2(i), \qquad I_3 = d_{22}\mathbf{E}\eta_i^2,$$
$$I_4 = 2d_{22}\frac{a_0}{1-a_1}\mathbf{E}x_i F_2(i), \qquad I_5 = 2d_{22}\frac{a_0}{1-a_1}\mathbf{E}x_i\eta_i,$$
$$I_6 = 2d_{22}a_1\mathbf{E}x_{i-1}F_2(i), \qquad I_7 = 2d_{22}a_1\mathbf{E}x_{i-1}\eta_i.$$

Let us estimate the summands $I_1, \ldots, I_7$. Via (3.2) we have

$$I_1 \le d_{22}\alpha_2 \sum_{k=-h}^{i-2} |a_{i-k}|\mathbf{E}x_k^2.$$

Note that $\mathbf{E}x_k x_l \xi_{j+1} = 0$ for $k, l \le j$ and $\mathbf{E}x_l^2 \xi_{j+1}^2 = \mathbf{E}x_l^2$ for $l \le j$. So,

$$
\begin{aligned}
|I_2| &= 2d_{22}\left|\mathbf{E}\sum_{j=0}^{i-3}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\left(\sum_{k=-h}^{j}a_{i-k}x_k+\sum_{k=j+1}^{i-2}a_{i-k}x_k\right)\right| \\
&= 2d_{22}\left|\mathbf{E}\sum_{j=0}^{i-3}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\left(\sum_{k=j+1}^{i-2}a_{i-k}x_k\right)\right| \\
&\le d_{22}\sum_{j=0}^{i-3}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-2}|a_{i-k}|\left(\mathbf{E}x_k^2+\mathbf{E}x_l^2\right) \\
&\le d_{22}\left(S_3\sum_{k=1}^{i-2}|a_{i-k}|\mathbf{E}x_k^2+\alpha_2\sum_{k=-h}^{i-3}\sum_{p=3}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\mathbf{E}x_k^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= d_{22}\sum_{j=0}^{i}\mathbf{E}\left(\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\right)^2 \le d_{22}\sum_{j=0}^{i}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=-h}^{j}\left|\sigma_{j-k}^{i-j}\right|\mathbf{E}x_k^2 \\
&\le d_{22}\sum_{k=-h}^{i}\sum_{j=k_m}^{i}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=0}^{\infty}\left|\sigma_l^{i-j}\right|\mathbf{E}x_k^2 \\
&= d_{22}\Bigg(\left|\sigma_0^0\right|\sum_{l=0}^{\infty}\left|\sigma_l^0\right|\mathbf{E}x_i^2+\sum_{p=0}^{1}\left|\sigma_{1-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\mathbf{E}x_{i-1}^2 \\
&\quad+\sum_{k=-h}^{i-2}\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\mathbf{E}x_k^2\Bigg).
\end{aligned}
$$

Similarly,

$$
|I_4| \le d_{22}\frac{|a_0|}{1-a_1}\sum_{l=-h}^{i-2}|a_{i-l}|\left(\mathbf{E}x_i^2+\mathbf{E}x_l^2\right) \le d_{22}\frac{|a_0|}{1-a_1}\left(\alpha_2\mathbf{E}x_i^2+\sum_{k=-h}^{i-2}|a_{i-k}|\mathbf{E}x_k^2\right)
$$

and

$$
\begin{aligned}
|I_5| &\le d_{22}\frac{|a_0|}{1-a_1}\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\left(\mathbf{E}x_i^2+\mathbf{E}x_l^2\right) \\
&\le d_{22}\frac{|a_0|}{1-a_1}\left(S_1\mathbf{E}x_i^2+\left|\sigma_0^1\right|\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\mathbf{E}x_k^2\right).
\end{aligned}
$$

Analogously we have

$$
|I_6| \le d_{22}|a_1|\sum_{l=-h}^{i-2}|a_{i-l}|\left(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2\right) \le d_{22}|a_1|\left(\alpha_2\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}|a_{i-k}|\mathbf{E}x_k^2\right)
$$

and

$$|I_7| \le d_{22}|a_1| \sum_{j=0}^{i-2} \sum_{l=-h}^{j} \left|\sigma_{j-l}^{i-j}\right| \left(\mathbf{E}x_{i-1}^2 + \mathbf{E}x_l^2\right)$$

$$= d_{22}|a_1| \left(S_2 \mathbf{E}x_{i-1}^2 + \sum_{k=-h}^{i-2} \sum_{p=2}^{i-k_m} \left|\sigma_{i-k-p}^{p}\right| \mathbf{E}x_k^2 \right).$$

As a result we obtain

$$\mathbf{E}\Delta V_{1i} \le (\gamma_0 - 1)\mathbf{E}x_i^2 + \gamma_1 \mathbf{E}x_{i-1}^2 + \sum_{k=-h}^{i-2} P_{ik} \mathbf{E}x_k^2,$$

where

$$\gamma_0 = d_{22} \left(\frac{|a_0|}{1-a_1} (\alpha_2 + S_1) + \left|\sigma_0^0\right| \sum_{l=0}^{\infty} \left|\sigma_l^0\right| \right),$$

$$\gamma_1 = d_{22} \left(|a_1|(\alpha_2 + S_2) + \frac{|a_0|}{1-a_1} \left|\sigma_0^1\right| + \sum_{p=0}^{1} \left|\sigma_{1-p}^p\right| \sum_{l=0}^{\infty} \left|\sigma_l^p\right| \right),$$

$$P_{ik} = d_{22} \Bigg[|a_{i-k}|(\alpha_2 + S_3) + \alpha_2 \sum_{p=3}^{i-k_m} \left|\sigma_{i-k-p}^p\right| + \sum_{p=0}^{i-k_m} \left|\sigma_{i-k-p}^p\right| \sum_{l=0}^{\infty} \left|\sigma_l^p\right|$$

$$+ \frac{|a_0|}{1-a_1} \left(|a_{i-k}| + \sum_{p=1}^{i-k_m} \left|\sigma_{i-k-p}^p\right| \right) + |a_1| \left(|a_{i-k}| + \sum_{p=2}^{i-k_m} \left|\sigma_{i-k-p}^p\right| \right) \Bigg].$$

It is easy to see that $P_{j+i,i}$ for $i \ge 0$ depends on j only. So,

$$\gamma_0 + \gamma_1 + \sum_{j=2}^{\infty} P_{j+i,i} = \left[\left[\alpha_2^2 + 2\alpha_2 S_3 + S_0 + 2|a_1|(\alpha_2 + S_2)\right](1-a_1)\right.$$

$$\left. + 2|a_0|(\alpha_2 + S_1)\right] \frac{1}{(1+a_1)[(1-a_1)^2 - a_0^2]}.$$

Let us suppose that

$$\frac{[\alpha_2^2 + 2\alpha_2 S_3 + S_0 + 2|a_1|(\alpha_2 + S_2)](1-a_1) + 2|a_0|(\alpha_2 + S_1)}{(1+a_1)[(1-a_1)^2 - a_0^2]} < 1. \qquad (3.13)$$

Using Theorem 1.2 we find that the inequalities (3.12) and (3.13) are a sufficient condition for asymptotic mean square stability of the trivial solution of (3.1).

In particular, for (2.1) we have $S_0 = \sigma^2$, $\alpha_2 = S_1 = S_2 = S_3 = 0$, and condition (3.13) gives (2.16).

Note that one can use other representations of (3.1) in the form (1.7) and obtain other sufficient conditions for asymptotic mean square stability of the trivial solution of (3.1).

3.4 Fourth Way of the Construction of the Lyapunov Functional

1. Represent the right-hand side of (3.1) in the form (1.7) with

$$\tau = 2, \qquad F_1(i, x_{i-2}, x_{i-1}, x_i) = a_0 x_i + a_1 x_{i-1} + a_2 x_{i-2},$$

$$F_2(i) = F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-3} a_{i-l} x_l,$$

$$F_3(i, x_{-h}, \dots, x_i) = G_1(i, j, x_{j-2}, x_{j-1}, x_j) = 0,$$

$$G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l, \quad j = 0, \dots, i, \ i = 0, 1, \dots.$$

2. In this case the auxiliary equation type of (1.8) is

$$y_{i+1} = a_0 y_i + a_1 y_{i-1} + a_2 y_{i-2}. \tag{3.14}$$

Introduce the vector $y(i) = (y_{i-2}, y_{i-1}, y_i)'$ and represent (3.14) in the form

$$y(i+1) = Ay(i), \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_2 & a_1 & a_0 \end{pmatrix}. \tag{3.15}$$

Consider now the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

where D is a symmetric matrix. The entries d_{ij} of the matrix D are defined by the following system of the equations:

$$\begin{aligned}
&a_2^2 d_{33} - d_{11} = 0, \\
&a_2 d_{13} + a_1 a_2 d_{33} - d_{12} = 0, \\
&a_2 d_{23} + a_0 a_2 d_{33} - d_{13} = 0, \\
&d_{11} + 2a_1 d_{13} + a_1^2 d_{33} - d_{22} = 0, \\
&d_{12} + a_0 d_{13} + a_0 a_1 d_{33} + (a_1 - 1) d_{23} = 0, \\
&d_{22} + 2a_0 d_{23} + \left(a_0^2 - 1\right) d_{33} = -1.
\end{aligned} \tag{3.17}$$

The solution of system (3.17) is

$$\begin{aligned}
d_{11} &= a_2^2 d_{33},\\
d_{12} &= \frac{a_2(1-a_1)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)}d_{33},\\
d_{13} &= \frac{a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)}d_{33},\\
d_{22} &= \left(a_1^2+a_2^2+\frac{2a_1a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)}\right)d_{33},\\
d_{23} &= \frac{(a_0+a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)}d_{33},\\
d_{33} &= \left(1-a_0^2-a_1^2-a_2^2-2a_0a_1a_2-\frac{2(a_0+a_2)(a_0+a_1a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)}\right)^{-1}.
\end{aligned} \tag{3.18}$$

Let us suppose that the solution D of matrix equation (3.16) with the matrix A defined in (3.15) is a positive semidefinite matrix with $d_{33} > 0$. In this case the function $v_i = y'(i)Dy(i)$ is the Lyapunov function for (3.14). In fact,

$$\begin{aligned}
\Delta v_i &= y'(i+1)Dy(i+1) - y'(i)Dy(i)\\
&= y'(i)\big[A'DA - D\big]y(i) = -y'(i)Uy(i) = -y_i^2.
\end{aligned}$$

3. Following the third step of the procedure the functional V_{1i} has to be chosen in the form $V_{1i} = x'(i)Dx(i)$, where $x(i) = (x_{i-2}, x_{i-1}, x_i)'$.
4. For calculating $\mathbf{E}\Delta V_{1i}$ represent (3.1) in the form

$$x(i+1) = Ax(i) + B(i), \tag{3.19}$$

$$B(i) = \big(0, 0, b(i)\big)', \quad b(i) = F_2(i) + \eta_i, \tag{3.20}$$

where the matrix A is defined in (3.15) and η_i in (3.2).

By virtue of (3.19) we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big]\\
&= \mathbf{E}\big[\big(Ax(i)+B(i)\big)'D\big(Ax(i)+B(i)\big) - x'(i)Dx(i)\big]\\
&= \mathbf{E}\big[-x_i^2 + 2B'(i)DAx(i) + B'(i)DB(i)\big].
\end{aligned}$$

Using (3.20) and the second and the third equations of system (3.17) it is easy to get

$$B'(i)DB(i) = d_{33}b^2(i),$$

$$B'(i)DAx(i) = \left(\frac{d_{13}}{a_2}x_i + \frac{d_{12}}{a_2}x_{i-1} + a_2d_{33}x_{i-2}\right)b(i).$$

From (3.18) it follows that the expressions $\frac{d_{12}}{a_2}$ and $\frac{d_{13}}{a_2}$ are definite also for $a_2 = 0$.

As a result, using (3.20) we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left[-x_i^2 + d_{33}b^2(i) + 2\left(\frac{d_{13}}{a_2}x_i + \frac{d_{12}}{a_2}x_{i-1} + a_2 d_{33}x_{i-2}\right)b(i)\right] \\ &= -\mathbf{E}x_i^2 + d_{33}\mathbf{E}b^2(i) + 2\frac{d_{13}}{a_2}\mathbf{E}x_i b(i) + 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}b(i) + 2a_2 d_{33}\mathbf{E}x_{i-2}b(i) \\ &= -\mathbf{E}x_i^2 + \sum_{k=1}^{9} I_k,\end{aligned}$$

where

$$\begin{aligned} I_1 &= d_{33}\mathbf{E}F_2^2(i), \qquad I_2 = d_{33}\mathbf{E}\eta_i^2, \qquad I_3 = 2d_{33}\mathbf{E}\eta_i F_2(i), \\ I_4 &= 2\frac{d_{13}}{a_2}\mathbf{E}x_i F_2(i), \qquad I_5 = 2\frac{d_{13}}{a_2}\mathbf{E}x_i\eta_i, \qquad I_6 = 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}F_2(i), \\ I_7 &= 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}\eta_i, \qquad I_8 = 2a_2 d_{33}\mathbf{E}x_{i-2}F_2(i), \qquad I_9 = 2a_2 d_{33}\mathbf{E}x_{i-2}\eta_i. \end{aligned}$$

Let us estimate the summands I_k, $k = 1, \dots, 9$. For I_1, I_2 we have

$$\begin{aligned} I_1 &\le d_{33}\sum_{l=-h}^{i-3}|a_{i-l}|\sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2 \le \alpha_3 d_{33}\sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2, \\ I_2 &= d_{33}\sum_{j=0}^{i}\mathbf{E}\left(\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\right)^2 \le d_{33}\sum_{j=0}^{i}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|\sum_{k=-h}^{j}|\sigma_{j-k}^{i-j}|\mathbf{E}x_k^2 \\ &\le d_{33}\sum_{k=-h}^{i}\sum_{j=k_m}^{i}|\sigma_{j-k}^{i-j}|\sum_{l=0}^{\infty}|\sigma_l^{i-j}|\mathbf{E}x_k^2 \\ &= d_{33}\Bigg(|\sigma_0^0|\sum_{l=0}^{\infty}|\sigma_l^0|\mathbf{E}x_i^2 + \sum_{p=0}^{1}|\sigma_{1-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_{i-1}^2 \\ &\quad + \sum_{p=0}^{2}|\sigma_{2-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=0}^{i-k_m}|\sigma_{i-p-k}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_k^2\Bigg). \end{aligned}$$

Using the properties of the conditional expectation, for I_3 we obtain

$$
\begin{aligned}
|I_3| &= 2d_{33}\left|\mathbf{E}\sum_{k=-h}^{i-3} a_{i-k}x_k \sum_{j=0}^{i}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\right| \\
&= 2d_{33}\left|\mathbf{E}\sum_{k=-h}^{i-3} a_{i-k}x_k \sum_{j=0}^{i-4}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\right| \\
&= 2d_{33}\left|\mathbf{E}\sum_{j=0}^{i-4}\left(\sum_{k=-h}^{j} a_{i-k}x_k + \sum_{k=j+1}^{i-3} a_{i-k}x_k\right)\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\right| \\
&= 2d_{33}\left|\mathbf{E}\sum_{j=0}^{i-4}\sum_{k=j+1}^{i-3} a_{i-k}x_k \sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\right| \\
&\le d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-3}|a_{i-k}|\left(\mathbf{E}x_l^2+\mathbf{E}x_k^2\right) \\
&= d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-3}|a_{i-k}|\mathbf{E}x_l^2 + d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\sum_{k=j+1}^{i-3}|a_{i-k}|\mathbf{E}x_k^2 \\
&\le \alpha_3 d_{33}\sum_{k=-h}^{i-4}\sum_{j=k_m}^{i-4}\left|\sigma_{j-k}^{i-j}\right|\mathbf{E}x_k^2 + d_{33}\sum_{k=1}^{i-3}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right||a_{i-k}|\mathbf{E}x_k^2 \\
&\le \alpha_3 d_{33}\sum_{k=-h}^{i-4}\sum_{p=4}^{i-k_m}\left|\sigma_{i-p-k}^{p}\right|\mathbf{E}x_k^2 + S_4 d_{33}\sum_{k=1}^{i-3}|a_{i-k}|\mathbf{E}x_k^2.
\end{aligned}
$$

Similar for the summands $I_4 - I_9$ we get

$$
|I_4| \le \left|\frac{d_{13}}{a_2}\right|\sum_{l=-h}^{i-3}|a_{i-l}|\left(\mathbf{E}x_i^2+\mathbf{E}x_l^2\right) \le \left|\frac{d_{13}}{a_2}\right|\left(\alpha_3\mathbf{E}x_i^2 + \sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2\right),
$$

$$
\begin{aligned}
|I_5| &= \left|2\frac{d_{13}}{a_2}\mathbf{E}x_i\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_{j+1}\right| \le \left|\frac{d_{13}}{a_2}\right|\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\left|\sigma_{j-l}^{i-j}\right|\left(\mathbf{E}x_i^2+\mathbf{E}x_l^2\right) \\
&\le \left|\frac{d_{13}}{a_2}\right|\left(S_1\mathbf{E}x_i^2 + \left|\sigma_0^1\right|\mathbf{E}x_{i-1}^2 + \sum_{p=1}^{2}\left|\sigma_{2-p}^{p}\right|\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=1}^{i-k_m}\left|\sigma_{i-p-k}^{p}\right|\mathbf{E}x_k^2\right),
\end{aligned}
$$

$$
|I_6| \le \left|\frac{d_{12}}{a_2}\right|\sum_{l=-h}^{i-3}|a_{i-l}\left(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2\right) \le \left|\frac{d_{12}}{a_2}\right|\left(\alpha_3\mathbf{E}x_{i-1}^2 + \sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2\right),
$$

$$|I_7| = \left| 2\frac{d_{12}}{a_2} \mathbf{E} x_{i-1} \sum_{j=0}^{i-2} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_{j+1} \right| \leq \left| \frac{d_{12}}{a_2} \right| \sum_{j=0}^{i-2} \sum_{l=-h}^{j} \left| \sigma_{j-l}^{i-j} \right| \left(\mathbf{E} x_{i-1}^2 + \mathbf{E} x_l^2 \right)$$

$$\leq \left| \frac{d_{12}}{a_2} \right| \left(S_2 \mathbf{E} x_{i-1}^2 + \left| \sigma_0^2 \right| \mathbf{E} x_{i-2}^2 + \sum_{k=-h}^{i-3} \sum_{p=2}^{i-k_m} \left| \sigma_{i-p-k}^p \right| \mathbf{E} x_k^2 \right),$$

$$|I_8| \leq |a_2| d_{33} \sum_{l=-h}^{i-3} |a_{i-l}| \left(\mathbf{E} x_{i-2}^2 + \mathbf{E} x_l^2 \right) \leq |a_2| d_{33} \left(\alpha_3 \mathbf{E} x_{i-2}^2 + \sum_{k=-h}^{i-3} |a_{i-k}| \mathbf{E} x_k^2 \right),$$

$$|I_9| = \left| 2 a_2 d_{33} \mathbf{E} x_{i-2} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_{j+1} \right| \leq |a_2| d_{33} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \left| \sigma_{j-l}^{i-j} \right| \left(\mathbf{E} x_{i-2}^2 + \mathbf{E} x_l^2 \right)$$

$$\leq |a_2| d_{33} \left(S_3 \mathbf{E} x_{i-2}^2 + \sum_{k=-h}^{i-3} \sum_{p=3}^{i-k_m} \left| \sigma_{i-p-k}^p \right| \mathbf{E} x_k^2 \right).$$

As a result we obtain

$$\mathbf{E} \Delta V_{1i} \leq (\gamma_0 - 1) \mathbf{E} x_i^2 + \gamma_1 \mathbf{E} x_{i-1}^2 + \gamma_2 \mathbf{E} x_{i-2}^2 + \sum_{k=-h}^{i-3} P_{ik} \mathbf{E} x_k^2, \tag{3.21}$$

where

$$\gamma_0 = \left| \frac{d_{13}}{a_2} \right| (\alpha_3 + S_1) + d_{33} \left| \sigma_0^0 \right| \sum_{l=0}^{\infty} \left| \sigma_l^0 \right|,$$

$$\gamma_1 = \left| \frac{d_{12}}{a_2} \right| (\alpha_3 + S_2) + \left| \frac{d_{13}}{a_2} \right| \left| \sigma_0^1 \right| + d_{33} \sum_{p=0}^{1} \left| \sigma_{1-p}^p \right| \sum_{l=0}^{\infty} \left| \sigma_l^p \right|,$$

$$\gamma_2 = d_{33} |a_2| (\alpha_3 + S_3) + \left| \frac{d_{12}}{a_2} \right| \left| \sigma_0^2 \right| + \left| \frac{d_{13}}{a_2} \right| \sum_{p=1}^{2} \left| \sigma_{2-p}^p \right| + d_{33} \sum_{p=0}^{2} \left| \sigma_{2-p}^p \right| \sum_{l=0}^{\infty} \left| \sigma_l^p \right|, \tag{3.22}$$

$$\gamma_3 = \sup_{i \in Z} \sum_{j=3}^{\infty} P_{j+i,i},$$

$$P_{ik} = d_{33} \left(S_4 |a_{i-k}| + \sum_{p=0}^{i-k_m} \left| \sigma_{i-p-k}^p \right| \sum_{l=0}^{\infty} \left| \sigma_l^p \right| \right)$$

$$+ \left| \frac{d_{13}}{a_2} \right| \left(|a_{i-k}| + \sum_{p=1}^{i-k_m} \left| \sigma_{i-p-k}^p \right| \right) + \left| \frac{d_{12}}{a_2} \right| \left(|a_{i-k}| + \sum_{p=2}^{i-k_m} \left| \sigma_{i-p-k}^p \right| \right)$$

$$+|a_2|d_{33}\left(|a_{i-k}|+\sum_{p=3}^{i-k_m}\left|\sigma_{i-p-k}^p\right|\right)$$

$$+\alpha_3 d_{33}\left(|a_{i-k}|+\sum_{p=4}^{i-k_m}\left|\sigma_{i-p-k}^p\right|\right). \tag{3.23}$$

Via (3.21) and Theorem 1.2 we find that if

$$\gamma_0+\gamma_1+\gamma_2+\gamma_3<1 \tag{3.24}$$

then the trivial solution of (3.1) is asymptotically mean square stable.

From (3.23) it follows that $P_{j+i,i}$ does not depend on i. So,

$$\begin{aligned}
\gamma_3 &= \sum_{j=3}^{\infty} P_{j+i,i} \\
&= \sum_{j=3}^{\infty}\Bigg[d_{33}\left(S_4|a_j|+\sum_{p=0}^{j}\left|\sigma_{j-p}^p\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right) \\
&\quad + \left|\frac{d_{13}}{a_2}\right|\left(|a_j|+\sum_{p=1}^{j}\left|\sigma_{j-p}^p\right|\right)+\left|\frac{d_{12}}{a_2}\right|\left(|a_j|+\sum_{p=2}^{j}\left|\sigma_{j-p}^p\right|\right) \\
&\quad + |a_2|d_{33}\left(|a_j|+\sum_{p=3}^{j}\left|\sigma_{j-p}^p\right|\right)+\alpha_3 d_{33}\left(|a_j|+\sum_{p=4}^{j}\left|\sigma_{j-p}^p\right|\right)\Bigg] \\
&= \alpha_3\left(d_{33}\big(|a_2|+\alpha_3+S_4\big)+\frac{|d_{12}|+|d_{13}|}{|a_2|}\right) \\
&\quad + d_{33}\sum_{j=3}^{\infty}\sum_{p=0}^{j}\left|\sigma_{j-p}^p\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|+\left|\frac{d_{13}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=1}^{j}\left|\sigma_{j-p}^p\right|+\left|\frac{d_{12}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=2}^{j}\left|\sigma_{j-p}^p\right| \\
&\quad + |a_2|d_{33}\sum_{j=3}^{\infty}\sum_{p=3}^{j}\left|\sigma_{j-p}^p\right|+\alpha_3 d_{33}\sum_{j=4}^{\infty}\sum_{p=4}^{j}\left|\sigma_{j-p}^p\right|.
\end{aligned}$$

From this and (3.22) it follows that

$$\begin{aligned}
&\gamma_0+\gamma_1+\gamma_2+\gamma_3 \\
&\quad = \left|\frac{d_{13}}{a_2}\right|(\alpha_3+S_1)+d_{33}\left|\sigma_0^0\right|\sum_{l=0}^{\infty}\left|\sigma_l^0\right| \\
&\qquad + \left|\frac{d_{12}}{a_2}\right|(\alpha_3+S_2)+\left|\frac{d_{13}}{a_2}\right|\left|\sigma_0^1\right|+d_{33}\sum_{p=0}^{1}\left|\sigma_{1-p}^p\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|
\end{aligned}$$

$$
\begin{aligned}
&+d_{33}|a_2|(\alpha_3+S_3)+\left|\frac{d_{12}}{a_2}\right||\sigma_0^2|+\left|\frac{d_{13}}{a_2}\right|\sum_{p=1}^{2}|\sigma_{2-p}^p|+d_{33}\sum_{p=0}^{2}|\sigma_{2-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p| \\
&+\alpha_3\left(d_{33}\big(|a_2|+\alpha_3+S_4\big)+\frac{|d_{12}|+|d_{13}|}{|a_2|}\right) \\
&+d_{33}\sum_{j=3}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|+\left|\frac{d_{13}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=1}^{j}|\sigma_{j-p}^p|+\left|\frac{d_{12}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=2}^{j}|\sigma_{j-p}^p| \\
&+|a_2|d_{33}\sum_{j=3}^{\infty}\sum_{p=3}^{j}|\sigma_{j-p}^p|+\alpha_3 d_{33}\sum_{j=4}^{\infty}\sum_{p=4}^{j}|\sigma_{j-p}^p| \\
=&\left|\frac{d_{13}}{a_2}\right|(\alpha_3+S_1)+\left|\frac{d_{12}}{a_2}\right|(\alpha_3+S_2)+|a_2|d_{33}(\alpha_3+S_3) \\
&+\alpha_3\left(d_{33}\big(|a_2|+\alpha_3+S_4\big)+\frac{|d_{12}|+|d_{13}|}{|a_2|}\right) \\
&+d_{33}\sum_{j=0}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|+\left|\frac{d_{13}}{a_2}\right|\sum_{j=1}^{\infty}\sum_{p=1}^{j}|\sigma_{j-p}^p|+\left|\frac{d_{12}}{a_2}\right|\sum_{j=2}^{\infty}\sum_{p=2}^{j}|\sigma_{j-p}^p| \\
&+|a_2|d_{33}\sum_{j=3}^{\infty}\sum_{p=3}^{j}|\sigma_{j-p}^p|+\alpha_3 d_{33}\sum_{j=4}^{\infty}\sum_{p=4}^{j}|\sigma_{j-p}^p|.
\end{aligned}
$$

Via (3.2) we have

$$
\sum_{j=0}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|=\sum_{p=0}^{\infty}\sum_{j=p}^{\infty}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|=\sum_{p=0}^{\infty}\left(\sum_{l=0}^{\infty}|\sigma_l^p|\right)^2=S_0,
$$

$$
\sum_{j=k}^{\infty}\sum_{p=k}^{j}|\sigma_{j-p}^p|=\sum_{p=k}^{\infty}\sum_{j=p}^{\infty}|\sigma_{j-p}^p|=\sum_{p=k}^{\infty}\sum_{l=0}^{\infty}|\sigma_l^p|=S_k,\quad k=1,2,3,4.
$$

Therefore

$$
\begin{aligned}
&\gamma_0+\gamma_1+\gamma_2+\gamma_3 \\
&\quad=\left|\frac{d_{13}}{a_2}\right|(\alpha_3+S_1)+\left|\frac{d_{12}}{a_2}\right|(\alpha_3+S_2)+|a_2|d_{33}(\alpha_3+S_3) \\
&\qquad+\alpha_3\left(d_{33}\big(|a_2|+\alpha_3+S_4\big)+\frac{|d_{12}|+|d_{13}|}{|a_2|}\right) \\
&\qquad+d_{33}S_0+\left|\frac{d_{13}}{a_2}\right|S_1+\left|\frac{d_{12}}{a_2}\right|S_2+|a_2|d_{33}S_3+\alpha_3 d_{33}S_4
\end{aligned}
$$

$$= 2\left|\frac{d_{13}}{a_2}\right|(\alpha_3 + S_1) + 2\left|\frac{d_{12}}{a_2}\right|(\alpha_3 + S_2)$$
$$+ d_{33}\left[S_0 + 2|a_2|(\alpha_3 + S_3) + 2\alpha_3 S_4 + \alpha_3^2\right].$$

Using representation (3.18) for d_{12}, d_{13}, we obtain

$$\begin{aligned}&\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3\\ &\quad = 2d_{33}\frac{|a_0 + a_1a_2|(\alpha_3 + S_1) + |(1 - a_1)(a_1 + a_0a_2)|(\alpha_3 + S_2)}{|1 - a_1 - a_2(a_0 + a_2)|}\\ &\qquad + d_{33}\left[S_0 + 2|a_2|(\alpha_3 + S_3) + 2\alpha_3 S_4 + \alpha_3^2\right].\end{aligned}$$

At last, using (3.18) for representation d_{33}, we rewrite condition (3.24) in the form

$$\begin{aligned}&2\frac{|a_0 + a_1a_2|(\alpha_3 + S_1) + |(1 - a_1)(a_1 + a_0a_2)|(\alpha_3 + S_2)}{1 - a_1 - a_2(a_0 + a_2)}\\ &\quad + S_0 + 2|a_2|(\alpha_3 + S_3) + 2\alpha_3 S_4 + \alpha_3^2\\ &\quad < 1 - a_0^2 - a_1^2 - a_2^2 - 2a_0a_1a_2 - \frac{2(a_0 + a_2)(a_0 + a_1a_2)(a_1 + a_0a_2)}{1 - a_1 - a_2(a_0 + a_2)},\\ &a_1 + a_2(a_0 + a_2) < 1.\end{aligned} \tag{3.25}$$

So, if the matrix D with entries (3.18) is a positive semidefinite one with $d_{33} > 0$ and inequality (3.25) holds then the trivial solution of (3.1) is asymptotically mean square stable.

3.5 One Generalization

Let us generalize the previous way of Lyapunov functional construction. Having for an object to simplify formal transformations we will consider the equation

$$x_{i+1} = \sum_{j=-h}^{i} a_{i-j}x_j + \sum_{j=-h}^{i} \sigma_{i-j}x_j\xi_{i+1} \tag{3.26}$$

that is more simple one than (3.1). Here a_j and σ_j are known constants, ξ_i is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

1. Represent (3.1) in the form (1.7) with

$$\tau = k \geq 0, \qquad F_1(i, x_{i-k}, \ldots, x_{i-1}, x_i) = \sum_{l=i-k}^{i} a_{i-l}x_l,$$

$$F_2(i) = F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-k-1} a_{i-l} x_l, \qquad F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i, j, x_{i-k}, \dots, x_j) = 0, \quad j = 0, \dots, i,$$

$$G_2(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i-1,$$

$$G_2(i, i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i} \sigma_{i-l} x_l, \quad i = 0, 1, \dots$$

2. In this case the auxiliary equation type of (1.8) is

$$y_{i+1} = \sum_{j=0}^{k} a_j y_{i-j}. \tag{3.27}$$

Introduce the vector $y(i) = (y_{i-k}, \dots, y_{i-1}, y_i)'$ and represent (3.27) in a matrix form

$$y(i+1) = Ay(i), \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \dots & a_1 & a_0 \end{pmatrix}. \tag{3.28}$$

Consider the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \tag{3.29}$$

and suppose that the solution D of this equation is a positive semidefinite symmetric matrix of dimension $k+1$ with $d_{k+1,k+1} > 0$. In this case the function $v_i = y_i' D y_i$ is a Lyapunov function for (3.28), i.e. it satisfies the conditions of Theorem 1.1; in particular, condition (1.9). In fact, using (3.28) and (3.29), we have $\Delta v_i = -y_i^2$.

3. Following the procedure the functional V_{1i} for (3.26) has been chosen in the form:

$$V_{1i} = x'(i) D x(i), \quad x(i) = (x_{i-k}, \dots, x_{i-1}, x_i)'.$$

Rewrite now (3.26) as follows

$$x(i+1) = Ax(i) + B(i), \tag{3.30}$$

where the matrix A is defined in (3.28),

$$B(i)=\big(0,\dots,0,b(i)\big)',\quad b(i)=F_2(i)+\eta_i,\ \eta_i=\sum_{l=0}^{i+h}\sigma_l x_{i-l}\xi_{i+1}. \tag{3.31}$$

Calculating ΔV_{1i}, by virtue of (3.30) and (3.29) we have

$$\begin{aligned}\Delta V_{1i}&=\big(Ax(i)+B(i)\big)'D\big(Ax(i)+B(i)\big)-x'(i)Dx(i)\\&=-x_i^2+B'(i)DB(i)+2B'(i)DAx(i).\end{aligned} \tag{3.32}$$

Put

$$\delta_0=\sum_{j=0}^{\infty}|\sigma_j|,\qquad \alpha_m=\sum_{j=m}^{\infty}|a_j|,\quad m=0,\dots,k+1. \tag{3.33}$$

Using (3.31), we have $B'(i)DB(i)=d_{k+1,k+1}b^2(i)$. So, via (3.31) and (3.33)

$$\begin{aligned}\mathbf{E}B'(i)DB(i)&=d_{k+1,k+1}\big(\mathbf{E}F^2(i)+\mathbf{E}\eta_i^2\big)\\&\le d_{k+1,k+1}\left[\alpha_{k+1}\sum_{l=k+1}^{i+h}|a_l|\mathbf{E}x_{i-l}^2+\delta_0\sum_{l=0}^{i+h}|\sigma_l|\mathbf{E}x_{i-l}^2\right]\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\mathbf{E}B'(i)DAx(i)&=\mathbf{E}b(i)\left[\sum_{l=1}^{k}d_{l,k+1}x_{i-k+l}+d_{k+1,k+1}\sum_{m=0}^{k}a_m x_{i-m}\right]\\&=\mathbf{E}b(i)\left[\sum_{m=0}^{k-1}(a_m d_{k+1,k+1}+d_{k-m,k+1})x_{i-m}+a_k d_{k+1,k+1}x_{i-k}\right]\\&=d_{k+1,k+1}\mathbf{E}\sum_{l=k+1}^{i+h}a_l x_{i-l}\sum_{m=0}^{k}Q_{km}x_{i-m},\end{aligned} \tag{3.35}$$

where

$$Q_{km}=a_m+\frac{d_{k-m,k+1}}{d_{k+1,k+1}},\quad m=0,\dots,k-1,\qquad Q_{kk}=a_k.$$

Putting

$$\beta_k=\sum_{m=0}^{k}|Q_{km}|=|a_k|+\sum_{m=0}^{k-1}\left|a_m+\frac{d_{k-m,k+1}}{d_{k+1,k+1}}\right| \tag{3.36}$$

and using (3.33), (3.35) and (3.36), we obtain

$$2\mathbf{E}B'(i)DAx(i) = 2d_{k+1,k+1}\sum_{m=0}^{k}\sum_{l=k+1}^{i+h} Q_{km}a_l\mathbf{E}x_{i-m}x_{i-l}$$
$$\leq d_{k+1,k+1}\left(\alpha_{k+1}\sum_{m=0}^{k}|Q_{km}|\mathbf{E}x_{i-m}^2 + \beta_k\sum_{l=k+1}^{i+h}|a_l|\mathbf{E}x_{i-l}^2\right). \tag{3.37}$$

Put now

$$R_{km} = \begin{cases} \alpha_{k+1}|Q_{km}| + \delta_0|\sigma_m|, & 0 \leq m \leq k \\ (\alpha_{k+1}+\beta_k)|a_m| + \delta_0|\sigma_m|, & m > k \end{cases}, \qquad \gamma_0 = \sum_{m=0}^{\infty} R_{km}. \tag{3.38}$$

Then from (3.32), (3.34) and (3.37) it follows that

$$\mathbf{E}\Delta V_{1i} \leq -\mathbf{E}x_i^2 + d_{k+1,k+1}\sum_{m=0}^{i+h} R_{km}\mathbf{E}x_{i-m}^2.$$

Therefore, if $d_{k+1,k+1}\gamma_0 < 1$ then the functional V_{1i} satisfies the conditions of Theorem 1.2. Using (3.33), (3.36) and (3.38), one can show that

$$\gamma_0 = \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k + \delta_0^2.$$

So, if

$$\alpha_{k+1} < \sqrt{\beta_k^2 + d_{k+1,k+1}^{-1} - \delta_0^2} - \beta_k. \tag{3.39}$$

then the trivial solution of (3.26) is asymptotically mean square stable.

Remark 3.1 If $a_j = 0$ for $j > k$ and matrix equation (3.29) has a positive semidefinite solution D with the conditions $d_{k+1,k+1} > 0$ and $\delta_0^2 < d_{k+1,k+1}^{-1}$ then the trivial solution of (3.26) is asymptotically mean square stable.

Remark 3.2 Suppose that in (3.26) $a_j = 0$ for $j > k$ and $\sigma_j = 0$ if $j \neq m$ for some m such that $0 \leq m \leq h$. In this case $\alpha_{k+1} = 0$, $\delta_0^2 = \sigma_m^2$ and from (3.32), (3.34) and (3.35) it follows that

$$\mathbf{E}\Delta V_{1i} = -\mathbf{E}x_i^2 + \sigma_m^2 d_{k+1,k+1}\mathbf{E}x_{i-m}^2.$$

Via Corollary 1.2 condition $\sigma_m^2 d_{k+1,k+1} < 1$ is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (3.26).

Remark 3.3 In the case $k = 0$ condition (3.39) takes the form $\alpha_0^2 + \delta_0^2 < 1$. In the case $k = 1$ condition (3.39) can be transform to the form

$$\alpha_0^2 + \delta_0^2 < 1 + \frac{2|a_0|}{1-a_1}\big(|a_1| - \alpha_0 a_1\big), \quad |a_1| < 1.$$

It is easy to see that this condition is not worse than the previous one. One can show also that for each $k = 1, 2, \ldots$ condition (3.39) is not worse than for the previous k.

Remark 3.4 From (3.2) it follows that for (3.26) $S_0 = \delta_0^2$ and $S_r = 0$ for $r > 0$. So, condition (3.39) for $k = 0$, $k = 1$ and $k = 2$ coincides with conditions (3.3), (3.13) and (3.25), respectively.

Example 3.1 Consider the equation

$$x_{i+1} = a_0 x_i + a_k x_{i-k} + \sigma x_{i-m}\xi_{i+1}, \quad k > 0. \tag{3.40}$$

Via (3.2) we have $\alpha_0 = |a_0| + |a_k|$, $S_0 = \sigma^2$, $S_1 = 0$. From (3.3) we get a sufficient condition for asymptotic mean square stability of the trivial solution of (3.40), which is more general than condition (2.2):

$$|a_0| + |a_k| < \sqrt{1 - \sigma^2}. \tag{3.41}$$

Via (3.2) we have also $\beta = a_0 + a_k$, $\alpha = k|a_k|$, $S_0 = \sigma^2$, $S_1 = S_2 = 0$. So, from condition (3.4) we obtain the sufficient condition for asymptotic mean square stability of the trivial solution of (3.40), which is a generalization of condition (2.5):

$$|a_0 + a_k| < \sqrt{1 - \sigma^2}, \quad \sigma^2 < (1 - a_0 - a_k)\big(1 + a_0 + a_k - 2k|a_k|\big).$$

At last for $k \geq 2$ we have $a_1 = 0$, $\alpha_2 = |a_k|$, $S_0 = \sigma^2$, $S_1 = S_2 = S_3 = 0$, and sufficient condition (3.41) for asymptotic mean square stability of the trivial solution of (3.40) follows from (3.13).

3.6 Investigation of Asymptotic Behavior via Characteristic Equation

In this section a deterministic linear Volterra difference equation is investigated. One known theorem on the asymptotic behavior of solution of this equation is considered where a stability criterion is derived via a positive root of the corresponding characteristic equation. Two new directions for further investigation are proposed. The first direction is connected with a weakening of the known stability criterion, the second one is connected with a consideration of not only positive but also of negative and complex roots of the characteristic equation. A lot of pictures with stability regions and trajectories of considered processes are presented for visual demonstration of the proposed directions.

3.6.1 Statement of the Problem

There is a series of papers (see, for example, [142–145, 203–206]) where a similar method is used for the investigation of the asymptotic behavior of solutions of difference equations [142, 144], differential equations [203, 204, 206], integro-differential equations [143, 145], difference equations with continuous time [205]. The basic assumption in this method is that the positive root of the corresponding characteristic equation satisfies a special sufficient condition for asymptotic stability of some auxiliary equation. By virtue of the stability conditions obtained above, here for the example of the difference Volterra equation we propose to extend the results of these investigations in two directions. Firstly it is shown that the basic assumption on the positive root of the corresponding characteristic equation can be essentially weakened using different conditions for the asymptotic stability. Besides of that it is shown that a consideration of negative and complex roots of the characteristic equation opens some new horizons for investigation. For visual demonstration of the proposed ideas a lot of pictures with numerical calculations of stability regions and trajectories of considered processes are presented.

Consider the Volterra difference equation

$$\Delta x_i = a x_i + \sum_{j=1}^{\infty} K_j x_{i-j}, \quad i \geq 0, \tag{3.42}$$

with the initial condition

$$x_j = \varphi_j, \quad j \leq 0. \tag{3.43}$$

Here $\Delta x_i = x_{i+1} - x_i$, a and K_j, $j = 1, 2, \ldots$, are real numbers. The equation

$$\lambda - 1 = a + \sum_{j=1}^{\infty} \lambda^{-j} K_j \tag{3.44}$$

is called the characteristic equation of the difference equation (3.42).

Theorem 3.1 *Let λ_0 be a positive root of the characteristic equation (3.44) with the property*

$$\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| < 1. \tag{3.45}$$

Then for any initial sequence φ_j, $j \leq 0$, the solution x_i of (3.42) and (3.43) satisfies the condition

$$\lim_{i \to \infty} \lambda_0^{-i} x_i = Q_{\lambda_0}(\varphi), \tag{3.46}$$

where

$$Q_{\lambda_0}(\varphi) = \frac{L_{\lambda_0}(\varphi)}{1 + \gamma_{\lambda_0}}, \quad \gamma_{\lambda_0} = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j,$$

$$L_{\lambda_0}(\varphi) = \varphi_0 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left(\sum_{r=-j}^{-1} \lambda_0^{-r} \varphi_r \right). \tag{3.47}$$

The proof of Theorem 3.1 follows from [123] where, in particular, it is shown that the sequence

$$z_i = \lambda_0^{-i} x_i - Q_{\lambda_0}(\varphi) \tag{3.48}$$

is a solution of the linear difference equation

$$z_i = -\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j} K_j\left(\sum_{r=i-j}^{i-1} z_r\right), \quad i > 0, \tag{3.49}$$

and by condition (3.45) z_i, defined by (3.48), converges to zero, which is equivalent to (3.46).

Two questions arise here.

Firstly, it is clear that the condition (3.45) is a sufficient condition for the asymptotic stability of the trivial solution of (3.49). But is condition (3.45) the best sufficient condition for asymptotic stability?

Secondly, why is only a positive root of (3.44) considered here? What situation have we in the case of a negative or complex root?

Below it is shown that condition (3.45) of Theorem 3.1 can be weakened and the negative and complex roots of (3.44) also can be useful for the investigation of the asymptotic behavior of the solution of (3.42) and (3.43).

3.6.2 Improvement of the Known Result

Rewrite (3.49) in the form

$$z_i = \sum_{l=1}^{\infty} a_l z_{i-l}, \quad i > 0, \; a_l = -\frac{1}{\lambda_0}\sum_{j=l}^{\infty}\lambda_0^{-j} K_j. \tag{3.50}$$

Equation (3.50) is a particular case of (3.1). Different sufficient conditions for asymptotic stability of the trivial solution of (3.50) were obtained above. In particular, from (3.3) it follows that if

$$\sum_{l=1}^{\infty} |a_l| < 1 \tag{3.51}$$

then the trivial solution of (3.50) is asymptotically stable. Condition (3.51) is weaker than (3.45). In fact,

$$\sum_{l=1}^{\infty}|a_l| \le \frac{1}{\lambda_0}\sum_{l=1}^{\infty}\sum_{j=l}^{\infty}\lambda_0^{-j}|K_j| = \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\sum_{l=1}^{j}\lambda_0^{-j}|K_j| = \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j} j|K_j| < 1. \tag{3.52}$$

Another sufficient condition for asymptotic stability of the trivial solution of difference Volterra equation (3.50) follows from (3.4): if $2\alpha - 1 < \beta < 1$, where

$$\alpha = \sum_{l=1}^{\infty} \left| \sum_{j=l+1}^{\infty} a_j \right|, \qquad \beta = \sum_{l=1}^{\infty} a_l,$$

then the trivial solution of (3.50) is asymptotically stable.

So, the following theorem is true.

Theorem 3.2 *Let λ_0 be a positive root of the characteristic equation (3.44) that satisfies the condition*

$$\frac{1}{\lambda_0} \sum_{l=1}^{\infty} \left| \sum_{j=l}^{\infty} \lambda_0^{-j} K_j \right| < 1 \tag{3.53}$$

or the condition

$$\begin{gathered} 2\alpha - 1 < \beta < 1, \\ \alpha = \frac{1}{\lambda_0} \sum_{l=1}^{\infty} \left| \sum_{j=l+1}^{\infty} (j-l)\lambda_0^{-j} K_j \right|, \ \beta = -\frac{1}{\lambda_0} \sum_{l=1}^{\infty} \sum_{j=l}^{\infty} \lambda_0^{-j} K_j. \end{gathered} \tag{3.54}$$

Then for any initial sequence φ_j, $j \le 0$, the solution of (3.42) and (3.43) satisfies conditions (3.46) and (3.47).

From (3.52) it follows that condition (3.53) is weaker than (3.45). To compare conditions (3.45), (3.53) and (3.54) consider the following example.

Example 3.2 Consider the difference equation

$$\Delta x_i = a x_i + K_1 x_{i-1} + K_2 x_{i-2}.$$

The auxiliary difference equation (3.50) in this case has the form

$$z_i = -\left(K_1 \lambda_0^{-2} + K_2 \lambda_0^{-3}\right) z_{i-1} - K_2 \lambda_0^{-3} z_{i-2}. \tag{3.55}$$

Conditions (3.45), (3.53) and (3.54) are, respectively,

$$|K_1|\lambda_0^{-2} + 2|K_2|\lambda_0^{-3} < 1, \tag{3.56}$$

$$\left|K_1\lambda_0^{-2} + K_2\lambda_0^{-3}\right| + |K_2|\lambda_0^{-3} < 1, \tag{3.57}$$

$$-1 < K_1\lambda_0^{-2} + 2K_2\lambda_0^{-3} < 1 - 2|K_2|\lambda_0^{-3}. \tag{3.58}$$

From (2.15) it follows that the necessary and sufficient condition for the asymptotic stability of the trivial solution of (3.55) is

$$\left|K_1\lambda_0^{-2} + K_2\lambda_0^{-3}\right| < 1 + K_2\lambda_0^{-3}, \quad |K_2|\lambda_0^{-3} < 1. \tag{3.59}$$

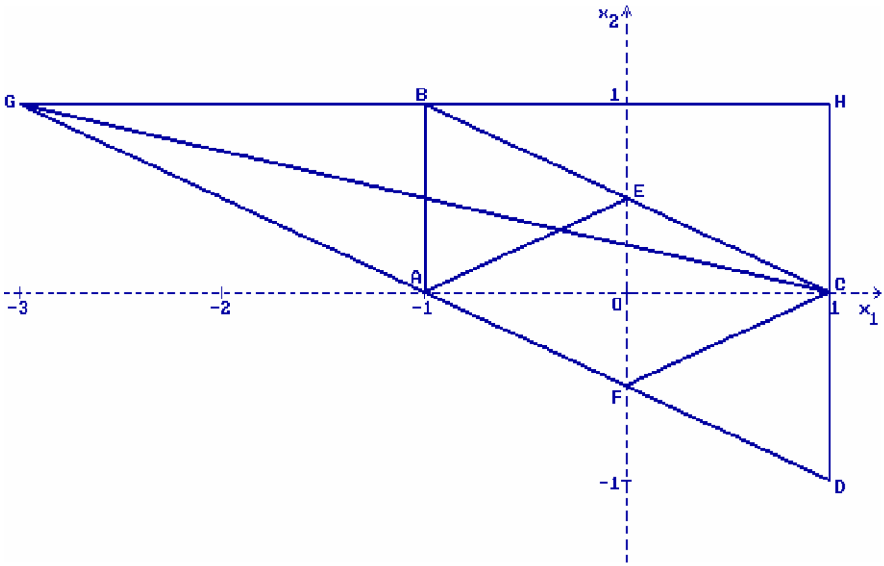


Fig. 3.1 Stability regions given by conditions: (3.56) (AECFA), (3.57) (ABCDA), (3.58) (GCDG) and (3.59) (GHDG)

One can see that condition (3.59) follows from each of the conditions (3.56)–(3.58). From each of these conditions it follows also that $1 + \gamma_{\lambda_0} = 1 + K_1\lambda_0^{-2} + 2K_2\lambda_0^{-3} > 0$, so $Q_{\lambda_0}(\phi)$ in (3.47) is defined.

In Fig. 3.1 stability regions for (3.55) are shown constructed by conditions (3.56) (region AECFA), (3.57) (region ABCDA), (3.58) (region GCDG) and (3.59) (region GHDG) in the space (x_1, x_2), where $x_1 = K_1\lambda_0^{-2}$, $x_2 = K_2\lambda_0^{-3}$.

3.6.3 Different Situations with Roots of the Characteristic Equation

To demonstrate the different situations of the use of not only positive but also of negative and complex roots of the characteristic equation (3.44), consider the simple difference equation

$$\begin{aligned} \Delta x_n &= ax_n + bx_{n-1}, \quad n = 0, 1, \dots, \\ x_j &= \varphi_j, \quad j = -1, 0. \end{aligned} \tag{3.60}$$

The corresponding characteristic equation is

$$\lambda - 1 = a + b\lambda^{-1}. \tag{3.61}$$

The following theorem deals with the behavior of the sequences x_n and $y_n = \lambda_0^{-n} x_n$, where x_n is a solution of (3.60), and λ_0 is a root of the characteristic equation (3.61).

Theorem 3.3 *We have the following four different situations with a solution of* (3.60).

(1) *If*

$$a+1 \neq 0 \quad \textit{and} \quad (a+1)^2 + 4b > 0 \tag{3.62}$$

then

$$\lim_{n\to\infty} y_n = Q_{\lambda_0}(\varphi), \tag{3.63}$$

where

$$Q_{\lambda_0}(\varphi) = \frac{L_{\lambda_0}(\varphi)}{1+\lambda_0^{-2}b}, \quad L_{\lambda_0}(\varphi) = \varphi_0 + \lambda_0^{-1} b\varphi_{-1}, \tag{3.64}$$

$$\lambda_0 = \frac{a+1}{2}\left(1+\sqrt{1+\frac{4b}{(a+1)^2}}\right). \tag{3.65}$$

(2) *If*

$$a+1 = 0 \quad \textit{and} \quad b > 0 \tag{3.66}$$

then $\lambda_0 = \pm\sqrt{b}$ *and*

$$y_{2k} = \varphi_0, \qquad y_{2k+1} = \lambda_0 \varphi_{-1}, \quad k = 0, 1, \ldots. \tag{3.67}$$

(3) *If*

$$a+1 \neq 0 \quad \textit{and} \quad (a+1)^2 + 4b = 0 \tag{3.68}$$

then

$$y_n = \varphi_0 + nL_{\lambda_0}(\varphi), \quad n = 0, 1, \ldots, \tag{3.69}$$

where $L_{\lambda_0}(\varphi)$ *is defined by* (3.64) *and* $\lambda_0 = \frac{1}{2}(a+1)$.

(4) *If*

$$(a+1)^2 + 4b < 0 \tag{3.70}$$

then

$$\left|y_n - Q_{\lambda_0}(\varphi)\right| = \left|\varphi_0 - Q_{\lambda_0}(\varphi)\right|, \quad n = 0, 1, \ldots, \tag{3.71}$$

where

$$Q_{\lambda_0}(\varphi) = \frac{\varphi_0}{2} \mp \mathrm{i}\frac{(a+1)\varphi_0 + 2b\varphi_{-1}}{2\sqrt{|(a+1)^2+4b|}}, \tag{3.72}$$

and λ_0 is one of the two conjugate complex roots

$$\lambda_0 = \frac{a+1 \pm \mathrm{i}\sqrt{|(a+1)^2+4b|}}{2}, \quad \mathrm{i} = \sqrt{-1}, \tag{3.73}$$

of the characteristic equation (3.61). *It means that the values of the process y_n are located in a complex plane on the circle with the center $Q_{\lambda_0}(\varphi)$ and the radius $r = |\varphi_0 - Q_{\lambda_0}(\varphi)|$. This circle includes the points* 0 *and* φ_0.

Proof

(1) Let us suppose that condition (3.62) holds. Put

$$z_n = y_n - Q_{\lambda_0}(\varphi), \quad y_n = \lambda_0^{-n} x_n, \tag{3.74}$$

where x_n is a solution of (3.60) and λ_0 is a root of the characteristic equation (3.61).

By condition (3.62), (3.61) has two real roots:

$$\lambda_{1,2} = \frac{a+1 \pm \sqrt{(a+1)^2+4b}}{2} = \frac{a+1}{2}\left(1 \pm \sqrt{1+\frac{4b}{(a+1)^2}}\right). \tag{3.75}$$

From (3.50) it follows that sequence (3.74) satisfies the equation

$$z_n = -\lambda_0^{-2} b z_{n-1}, \quad n = 0, 1, \ldots. \tag{3.76}$$

The necessary and sufficient condition for the asymptotic stability of the trivial solution of (3.76) is

$$\left|\lambda_0^{-2} b\right| < 1. \tag{3.77}$$

From (3.61) it follows that condition (3.77) is equivalent to $|1-(a+1)\lambda_0^{-1}| < 1$ or

$$\lambda_0 (a+1)^{-1} > \frac{1}{2}. \tag{3.78}$$

It is easy to see that from two roots (3.75) of (3.61), root (3.65) only satisfies condition (3.78). So (3.63) is proven.

(2) By conditions (3.66) from (3.61) it follows that $\lambda_0^{-2} b = 1$. Equation (3.76) takes the form $z_n = -z_{n-1}$. Therefore, $z_n = (-1)^n z_0$, $n = 1, 2, \ldots$. Via (3.74) and (3.64) from this we have

$$\begin{aligned} y_n &= Q_{\lambda_0}(\varphi) + (-1)^n \left[\varphi_0 - Q_{\lambda_0}(\varphi)\right] \\ &= (-1)^n \varphi_0 + \frac{1}{2}\left[1-(-1)^n\right]\left[\varphi_0 + \left(\lambda_0^{-2} b\right)\lambda_0 \varphi_{-1}\right] \\ &= \frac{1}{2}\left[1+(-1)^n\right]\varphi_0 + \frac{\lambda_0}{2}\left[1-(-1)^n\right]\varphi_{-1} \end{aligned}$$

which is equivalent to (3.67).

(3) By condition (3.68) the solution of (3.61) is $\lambda_0 = \frac{1}{2}(a+1)$. From this and (3.68) it follows that $1 + \lambda_0^{-2} b = 0$ and, therefore, $Q_{\lambda_0}(\phi)$ in (3.64) is undefined. It means that the sequence (3.74) undefined too. Using $y_j = \lambda_0^{-j} x_j$, $j = 0, 1, \ldots$, (3.60), (3.61) and $\lambda_0^{-1} b = -\lambda_0$, we have

$$\begin{aligned}\Delta x_j - a x_j - b x_{j-1} &= \Delta\big(\lambda_0^j y_j\big) - a\lambda_0^j y_j - b\lambda_0^{j-1} y_{j-1}\\ &= \lambda_0^j\big[\lambda_0 \Delta y_j + (\lambda_0 - 1 - a) y_j - b\lambda_0^{-1} y_{j-1}\big]\\ &= \lambda_0^{j+1}[\Delta y_j - \Delta y_{j-1}] = 0.\end{aligned}$$

From this via (3.64) it follows that

$$\Delta y_j = \Delta y_{j-1} = y_0 - y_{-1} = x_0 - \lambda_0 x_{-1} = \varphi_0 + \lambda_0^{-1} b \varphi_{-1} = L_{\lambda_0}(\varphi)$$

or $y_j = y_{j-1} + L_{\lambda_0}(\phi)$. Summing this equality with respect to $j = 1, 2, \ldots, n$, we obtain (3.69).

(4) Let us suppose now that condition (3.70) holds. Then the conjugate complex roots of (3.61) are defined by (3.73) and satisfy the condition $|\lambda_0|^2 = -b = |b|$ or $|\lambda_0^{-2} b| = 1$. From (3.76) it follows that the process (3.74) satisfies the equation $|z_n| = |z_{n-1}|$ or $|z_n| = |z_0|$. It is equivalent to (3.71). Now it is enough to show that $Q_{\lambda_0}(\varphi)$ defined by (3.64) equals $Q_{\lambda_0}(\varphi)$, defined by (3.72). In fact, putting $\delta = \sqrt{|(a+1)^2 + 4b|}$, from (3.73) we obtain $2\lambda_0 - (a+1) = \pm i\delta$. Using (3.64), (3.61) and (3.73), one can transform $Q_{\lambda_0}(\varphi)$ in the following way:

$$\begin{aligned}Q_{\lambda_0}(\varphi) &= \frac{L_{\lambda_0}(\varphi)}{2 - \lambda_0^{-1}(a+1)} = \frac{\lambda_0 L_{\lambda_0}(\varphi)}{2\lambda_0 - (a+1)} = \frac{\lambda_0 L_{\lambda_0}(\varphi)}{\pm i\delta} = \frac{2i\lambda_0 L_{\lambda_0}(\varphi)}{\mp 2\delta}\\ &= \frac{i((a+1)\varphi_0 \pm i\delta\varphi_0 + 2b\varphi_{-1})}{\mp 2\delta} = \frac{\varphi_0}{2} \mp i\frac{(a+1)\varphi_0 + 2b\varphi_{-1}}{2\delta}.\end{aligned}$$

The theorem is proven. □

The four regions described in Theorem 3.3 are shown in Fig. 3.2: (1) at the left of the curve $KLMK$ and from the right of the curve $KLNK$; (2) the line KL; (3) the curve $MLNM$; 4) under the curve $MLNM$. The point L with the coordinates $a = -1$, $b = 0$ is excluded, since in this point $\lambda_0 = 0$. The inside of the triangle ABC is the region of asymptotic stability of the trivial solution of (3.60).

Below in Figs. 3.3–3.6 the first situation of Theorem 3.3 is shown.

In Fig. 3.3 the trajectories of the processes x_n and y_n are shown in the point D (shown in Fig. 3.2) with the coordinates $a = -1.5$, $b = 0.65$. Here $\varphi_{-1} = 2$, $\varphi_0 = 0.5$, $\lambda_0 = -1.094$ (a negative root). The point P does not belong to the stability region (the triangle ABC) of the trivial solution of (3.60), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) quickly enough converges to $Q_{\lambda_0}(\varphi) = -0.446$.

In Fig. 3.4 the similar trajectories of the processes x_n and y_n are shown in the point E (Fig. 3.2). Here $a = -0.5$, $b = 0.65$, $\varphi_{-1} = -2$, $\varphi_0 = 2.5$, $\lambda_0 = 1.094$

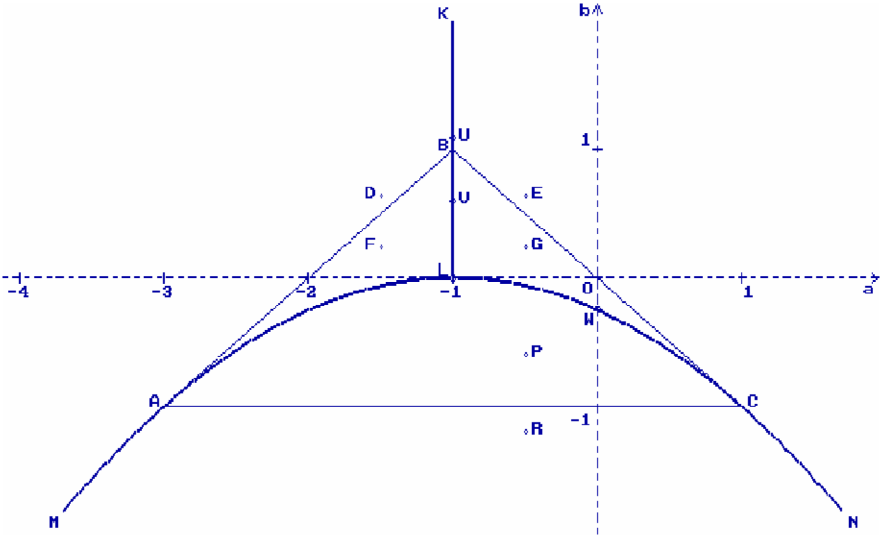


Fig. 3.2 Regions with different behaviors of x_n and y_n

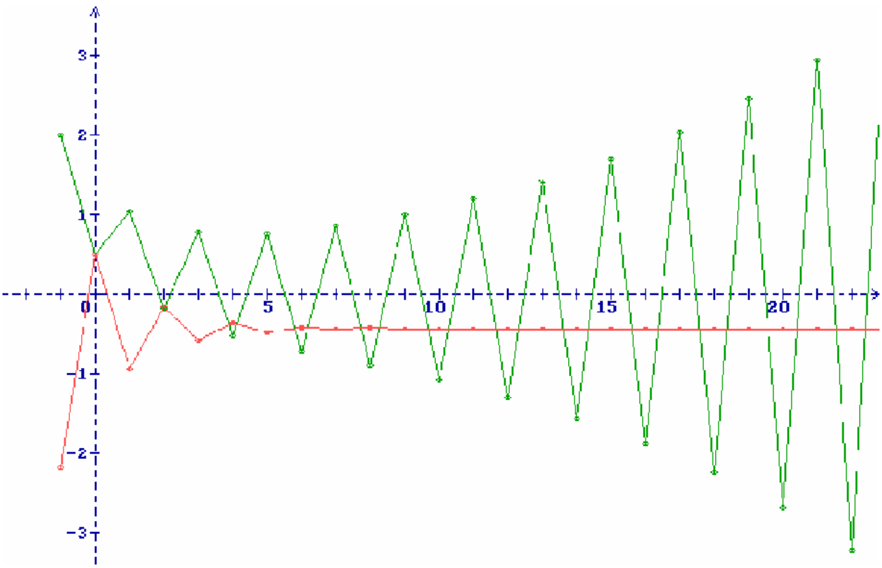


Fig. 3.3 Behavior of x_n and y_n in the point D

(a positive root), the process x_n (green) goes to $+\infty$, the process y_n (red) quickly converges to $Q_{\lambda_0}(\varphi) = 0.850$.

In Fig. 3.5 the trajectories of the processes x_n and y_n are shown in the point F (Fig. 3.2) with the coordinates $a = -1.5$, $b = 0.25$. Here $\varphi_{-1} = 3$, $\varphi_0 = -1.5$,

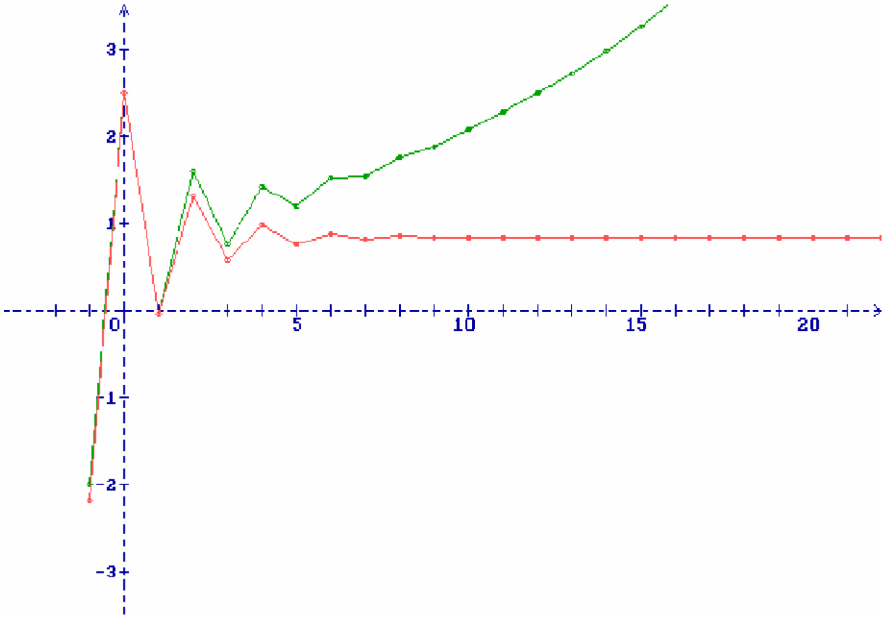


Fig. 3.4 Behavior of x_n and y_n in the point E

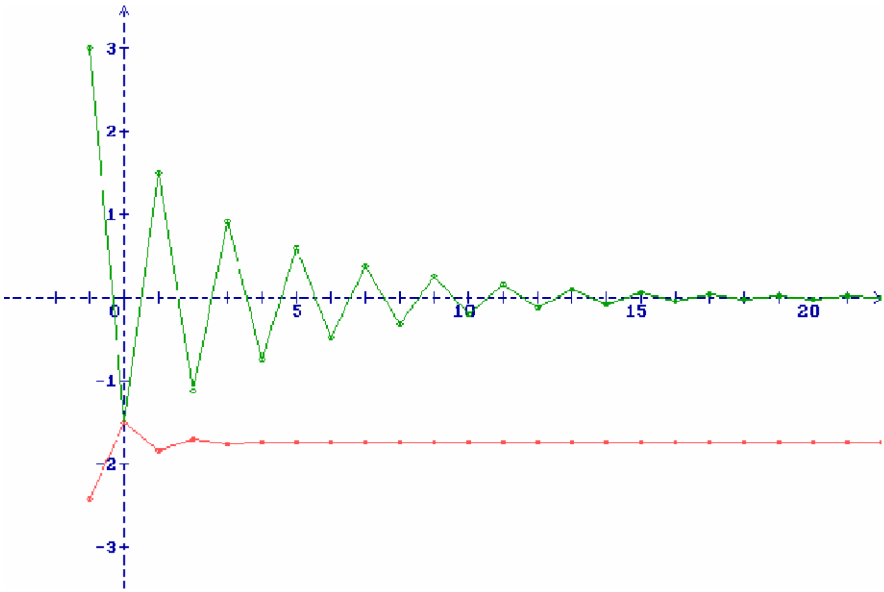


Fig. 3.5 Behavior of x_n and y_n in the point F

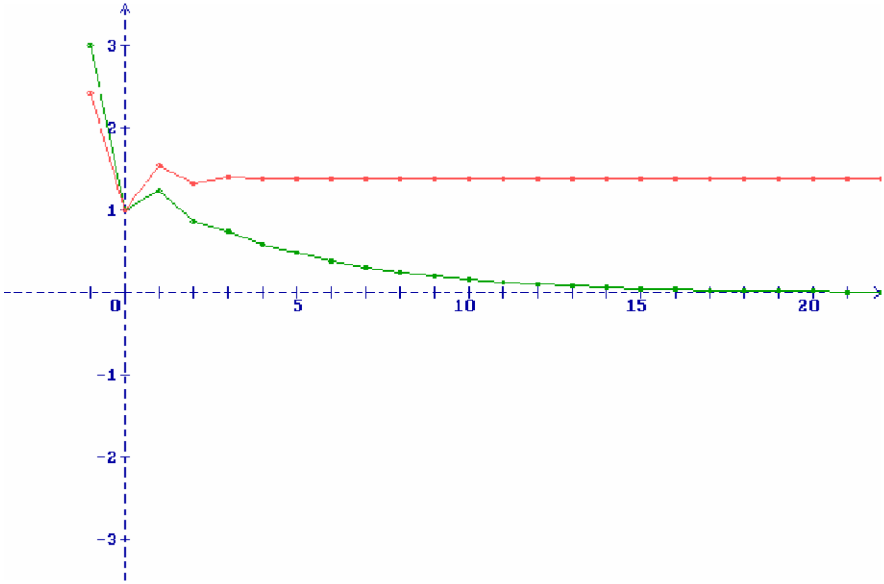


Fig. 3.6 Behavior of x_n and y_n in the point G

$\lambda_0 = -0.809$ (a negative root). The point F belongs to the stability region (the triangle ABC) of the trivial solution of (3.60), so the process x_n (green) converges to zero. The process y_n (red) quickly converges to $Q_{\lambda_0}(\varphi) = -1.756$.

In Fig. 3.6 similar trajectories of the processes x_n and y_n are shown in the point G (Fig. 3.2). Here $a = -0.5$, $b = 0.25$, $\varphi_{-1} = 3$, $\varphi_0 = 1$, $\lambda_0 = 0.809$ (a positive root), the process x_n (green) converges to zero, the process y_n (red) quickly converges to $Q_{\lambda_0}(\varphi) = 1.394$.

In Figs. 3.7 and 3.8 the second situation from Theorem 3.3 is shown.

In Fig. 3.7 the trajectories of the processes x_n and y_n are shown in the point U (Fig. 3.2) with the coordinates $a = -1$, $b = 1.1$. Here $\varphi_{-1} = 1.5$, $\varphi_0 = -1$, $\lambda_0 = -1.049$ (a negative root). The point U does not belong to the stability region (the triangle ABC) of the trivial solution of (3.43), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) has two values: $\varphi_0 = -1$ and $\lambda_0\varphi_{-1} = -1.573$.

In Fig. 3.8 the similar trajectories of the processes x_n and y_n are shown in the point V (Fig. 3.2). Here $a = -1$, $b = 0.6$, $\varphi_{-1} = 1.5$, $\varphi_0 = -1$, $\lambda_0 = 0.775$ (a positive root), the process x_n (green) converges to zero, the process y_n (red) has two values: $\varphi_0 = -1$ and $\lambda_0\varphi_{-1} = 1.162$.

In Figs. 3.9 and 3.10 the third situation from Theorem 3.3 is shown.

In Fig. 3.9 the trajectories of the processes x_n and y_n are shown in the point W (Fig. 3.2) with the coordinates $a = 0$, $b = -0.25$. Here $\varphi_{-1} = 3.5$, $\varphi_0 = 1.6$, $\lambda_0 = 0.5$ (a positive root), $L_{\lambda_0}(\varphi) = -0.15$. The point W belongs to the stability region (the triangle ABC) of the trivial solution of (3.60), so the process x_n (green) converges to zero. The process y_n (red) is a straight line.

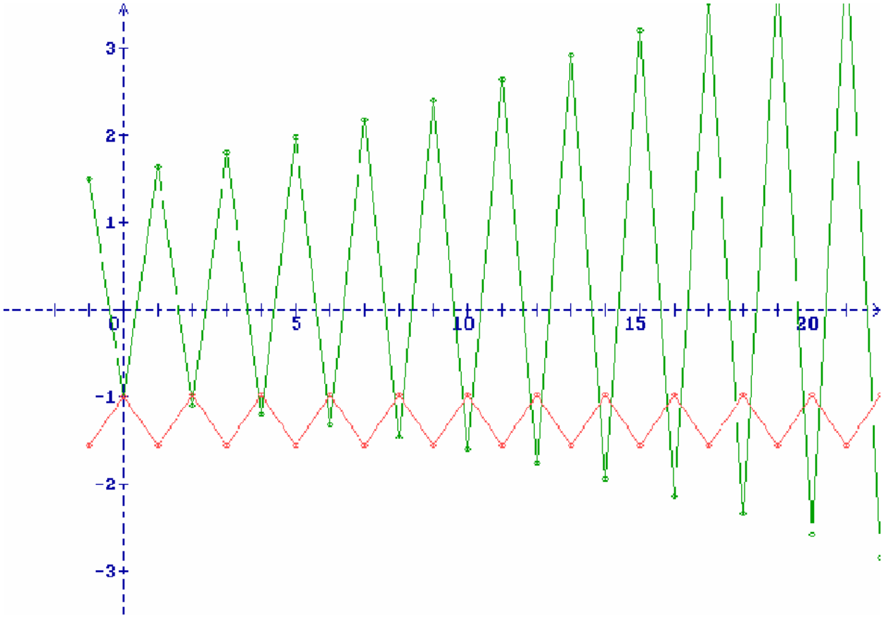


Fig. 3.7 Behavior of x_n and y_n in the point U

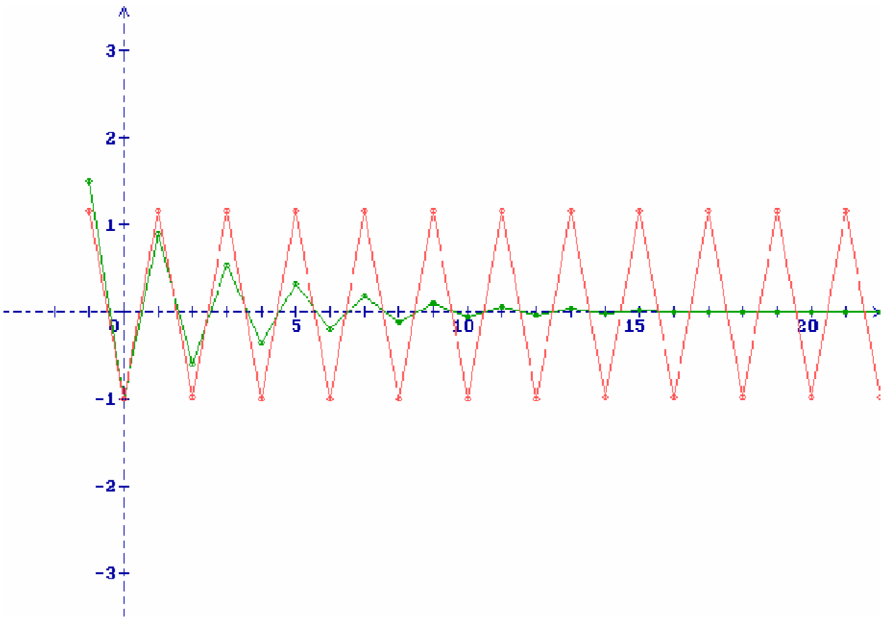


Fig. 3.8 Behavior of x_n and y_n in the point V

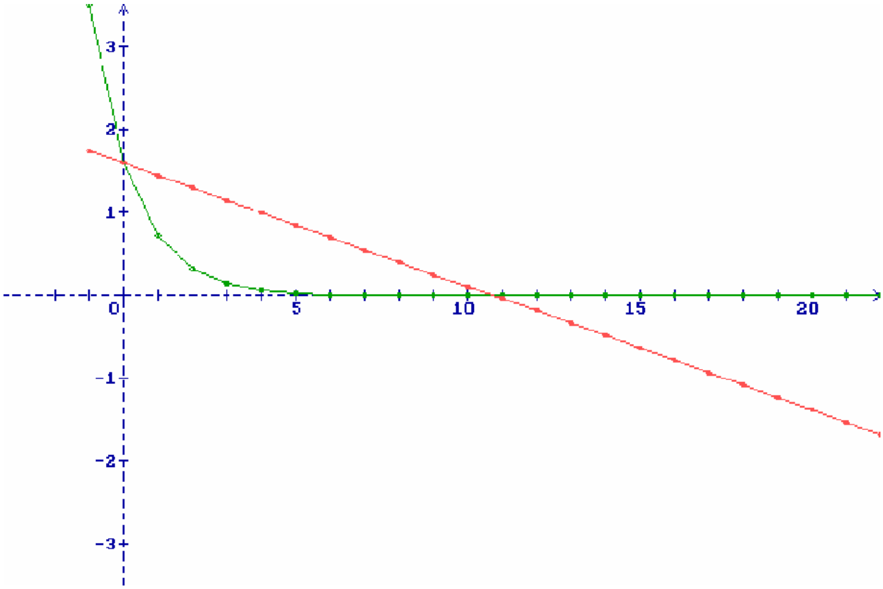


Fig. 3.9 Behavior of x_n and y_n in the point W

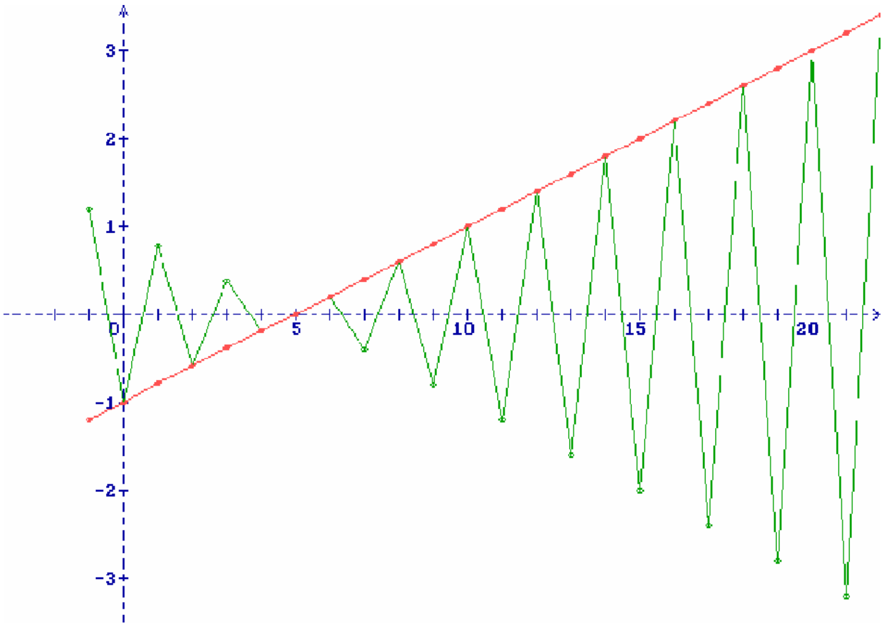
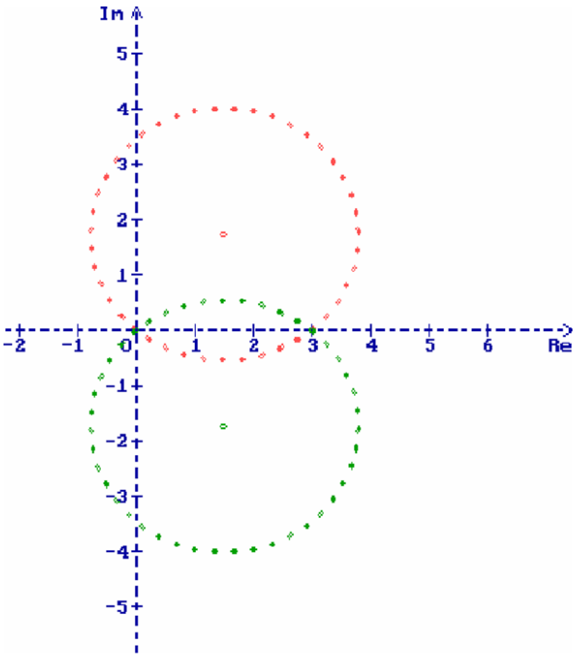
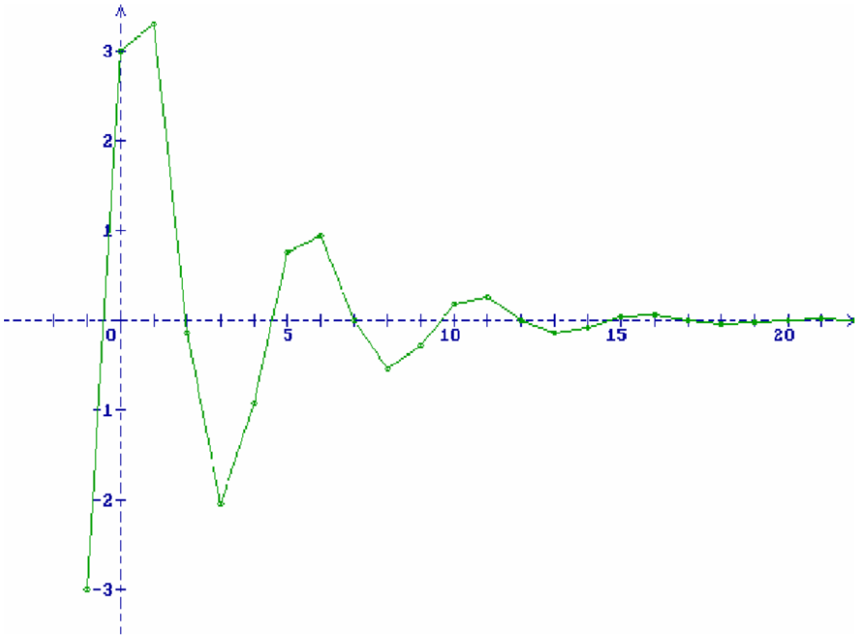


Fig. 3.10 Behavior of x_n and y_n in the point A

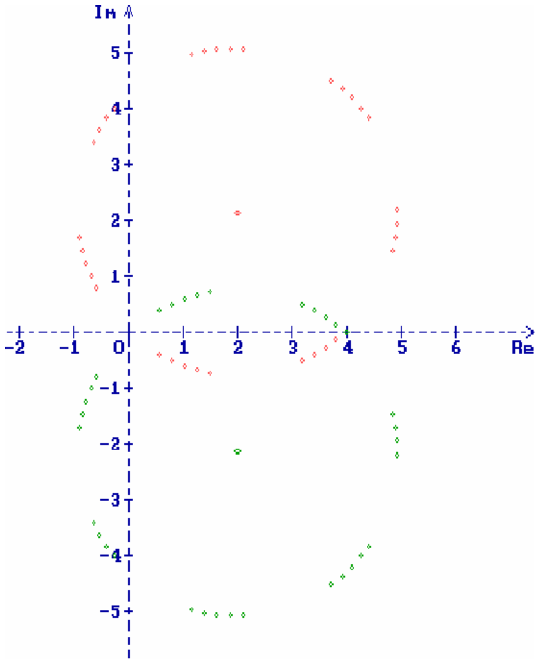


(a)

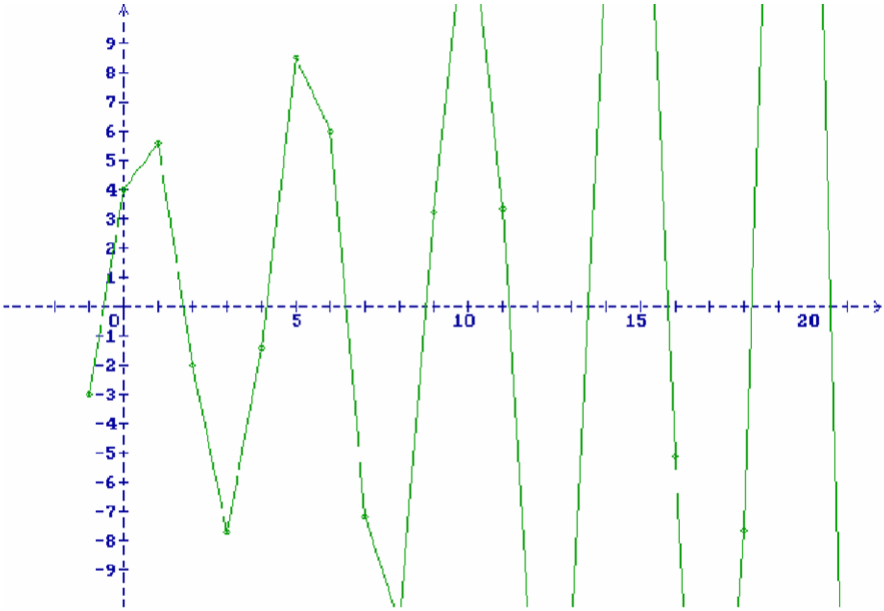


(b)

Fig. 3.11 **a** Behavior of y_n in the point P. **b** Behavior of x_n in the point P



(a)



(b)

Fig. 3.12 **a** Behavior of y_n in the point R. **b** Behavior of x_n in the point R

In Fig. 3.10 the trajectories of the processes x_n and y_n are shown in the point A (Fig. 3.2) with the coordinates $a = -3$, $b = -1$. Here $\varphi_{-1} = 1.2$, $\varphi_0 = -1$, $\lambda_0 = -1$ (a negative root), $L_{\lambda_0}(\varphi) = 0.2$. The point A does not belong to the stability region (the triangle ABC) of the trivial solution of (3.7), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) is a straight line.

In Figs. 3.11 and 3.12 the fourth situation from Theorem 3.3 is shown.

In Fig. 3.11a the trajectory of the complex process y_n is shown in the point P (Fig. 3.2) with the coordinates $a = -0.5$, $b = -0.6$. Here $\varphi_{-1} = -3$, $\varphi_0 = 3$. One can see that the values of the process y_n are located in the complex plane on the circle with radius $r = 2.297$ and the center $Q_{\lambda_0}(\varphi) = 1.5 - \mathrm{i}1.739$ (green) if $\lambda_0 = 0.25 + \mathrm{i}0.733$ and $Q_{\lambda_0}(\phi) = 1.5 + \mathrm{i}1.739$ (red) if $\lambda_0 = 0.25 - \mathrm{i}0.733$. In Fig. 3.1b the trajectory of the process x_n is shown in the same point P (Fig. 3.2). This point belongs to the stability region (the triangle ABC) of the trivial solution of (3.60), so the process x_n converges to zero.

In Fig. 3.12a the trajectory of the complex process y_n is shown in the point R (Fig. 3.2) with the coordinates $a = -0.5$, $b = -1.2$. Here $\varphi_{-1} = -3$, $\varphi_0 = 4$. One can see that the values of the process y_n are located in the complex plane on the circle with radius $r = 2.941$ and the center $Q_{\lambda_0}(\varphi) = 2 - \mathrm{i}2.157$ (green) if $\lambda_0 = 0.25 + \mathrm{i}1.067$ and $Q_{\lambda_0}(\varphi) = 2 + \mathrm{i}2.157$ (red) if $\lambda_0 = 0.25 - \mathrm{i}1.067$. In Fig. 3.12b the trajectory of the process x_n is shown in the same point R (Fig. 3.2). This point does not belong to the stability region (the triangle ABC) of the trivial solution of (3.60), so the process x_n goes to $\pm\infty$.

Chapter 4
Linear Equations with Nonstationary Coefficients

In this chapter via the procedure of the construction of Lyapunov functionals different types of stability conditions are obtained for linear equations with nonstationary coefficients.

4.1 First Way of the Construction of the Lyapunov Functional

Consider the linear equation

$$\begin{aligned} x_{i+1} &= \sum_{l=-h}^{i} a_{il}x_l + \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^{i} x_l \xi_{j+1}, \quad i \in Z, \\ x_i &= \varphi_i, \quad i \in Z_0. \end{aligned} \tag{4.1}$$

Here a_{il} and σ_{jl}^{i} are known constants, ξ_i, $i \in Z$, is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

Put also

$$\alpha_0 = \sup_{i\in Z} \sum_{l=-h}^{i} |a_{il}|, \qquad S_1 = \sup_{i\in Z} \sum_{j=0}^{i-1}\sum_{k=-h}^{j} \left|\sigma_{jk}^{i}\right|, \qquad \eta_i = \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^{i} x_l \xi_{j+1}. \tag{4.2}$$

1. Represent the right-hand side of (4.1) in the form (1.7), where $\tau = 0$,

$$F_1(i, x_i) = a_{ii}x_i, \qquad F_2(i, x_{-h}, \ldots, x_i) = \sum_{l=-h}^{i-1} a_{il}x_l,$$

$$F_3(i, x_{-h}, \ldots, x_i) = 0, \qquad G_1(i, j, x_j) = 0,$$

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_4,

$$G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{jl}^i x_l, \quad j = 0, \dots, i, \ i = 0, 1, \dots .$$

2. The auxiliary equation (1.8) in this case is $y_{i+1} = a_{ii} y_i$. Below, it is supposed that $\sup_{i \in Z} |a_{ii}| < 1$. By this condition the function $v_i = y_i^2$ is a Lyapunov function for the auxiliary equation, since $\Delta v_i = (a_{ii}^2 - 1) y_i^2$.
3. The functional V_{1i} has to be chosen in the form $V_{1i} = x_i^2$.
4. Calculating $\mathbf{E}\Delta V_{1i}$ we obtain

$$\mathbf{E}\Delta V_{1i} = \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = -\mathbf{E}x_i^2 + \mathbf{E}\left(\sum_{l=-h}^{i} a_{il} x_l + \eta_i\right)^2 = -\mathbf{E}x_i^2 + \sum_{k=1}^{3} I_k,$$

where

$$I_1 = \mathbf{E}\left(\sum_{l=-h}^{i} a_{il} x_l\right)^2, \qquad I_2 = 2\mathbf{E}\eta_i \sum_{l=-h}^{i} a_{il} x_l, \qquad I_3 = \mathbf{E}\eta_i^2.$$

Then via (4.2)

$$I_1 \le \alpha_0 \sum_{l=-h}^{i} |a_{il}| \mathbf{E}x_l^2,$$

$$|I_2| = 2\left|\mathbf{E}\eta_i\left(\sum_{l=-h}^{j} a_{il} x_l + \sum_{l=j+1}^{i} a_{il} x_l\right)\right| = 2\left|\mathbf{E}\sum_{j=0}^{i-1}\sum_{k=-h}^{j} \sigma_{jk}^i x_k \xi_{j+1} \sum_{l=j+1}^{i} a_{il} x_l\right|$$
$$\le \sum_{j=0}^{i-1}\sum_{k=-h}^{j}\sum_{l=j+1}^{i} \left|\sigma_{jk}^i\right| |a_{il}| \big(\mathbf{E}x_k^2 + \mathbf{E}x_l^2\big) = \sum_{k=-h}^{i-1} \alpha_{ik} \mathbf{E}x_k^2 + \sum_{l=1}^{i} \beta_{il} \mathbf{E}x_l^2,$$

where

$$\alpha_{ik} = \sum_{j=k_m}^{i-1} \left|\sigma_{jk}^i\right| \sum_{l=j+1}^{i} |a_{il}| \le \alpha_0 \sum_{j=k_m}^{i-1} \left|\sigma_{jk}^i\right|,$$

$$\beta_{il} = |a_{il}| \sum_{j=0}^{l-1}\sum_{k=-h}^{j} \left|\sigma_{jk}^i\right| \le S_1 |a_{il}|.$$

Similar

$$I_3 = \mathbf{E}\sum_{j=0}^{i}\left(\sum_{l=-h}^{j} \sigma_{jl}^i x_l\right)^2 \le \sum_{j=0}^{i}\sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \sum_{k=-h}^{j} \left|\sigma_{jk}^i\right| \mathbf{E}x_k^2 = \sum_{k=-h}^{i} \gamma_{ik} \mathbf{E}x_k^2,$$

where

$$\gamma_{ik} = \sum_{j=k_m}^{i} |\sigma_{jk}^i| \sum_{l=-h}^{j} |\sigma_{jl}^i|.$$

Therefore,

$$\mathbf{E}\Delta V_{1i} \le -\mathbf{E}x_i^2 + \sum_{k=-h}^{i} A_{ik}\mathbf{E}x_k^2,$$

where

$$A_{ik} = (\alpha_0 + S_1)|a_{ik}| + \alpha_0 \sum_{j=k_m}^{i-1} |\sigma_{jk}^i| + \gamma_{ik}.$$

Put

$$\hat{\alpha} = \sup_{i\in Z} \sum_{j=0}^{\infty} |a_{j+i,i}|, \qquad \hat{S}_1 = \sup_{i\in Z} \sum_{j=0}^{\infty} \sum_{l=i}^{j+i-1} |\sigma_{li}^{j+i}|,$$
$$S_0 = \sup_{i\in Z} \sum_{j=0}^{\infty} \sum_{m=i}^{j+i} |\sigma_{mi}^{j+i}| \sum_{l=-h}^{m} |\sigma_{ml}^{j+i}|. \tag{4.3}$$

Then

$$\sum_{j=0}^{\infty} A_{j+i,i} = (\alpha_0 + S_1) \sum_{j=0}^{\infty} |a_{j+i,i}| + \alpha_0 \sum_{j=0}^{\infty} \sum_{l=i}^{j+i-1} |\sigma_{li}^{j+i}| + \sum_{j=0}^{\infty} \gamma_{j+i,i}$$
$$\le (\alpha_0 + S_1)\hat{\alpha}_0 + \alpha_0 \hat{S}_1 + S_0.$$

From Theorem 1.2 it follows that the inequality

$$\alpha_0\hat{\alpha}_0 + \hat{\alpha}_0 S_1 + \alpha_0 \hat{S}_1 + S_0 < 1 \tag{4.4}$$

is a sufficient condition for the asymptotic mean square stability of the trivial solution of (4.1).

Remark 4.1 In the stationary case, i.e. $a_{jk} = a_{j-k}$, $\sigma_{jk}^i = \sigma_{j-k}^{i-j}$, condition (4.4) coincides with (3.3). In fact, via (4.2) and (4.3) we have

$$\alpha_0 = \sup_{i\in Z} \sum_{l=-h}^{i} |a_{il}| = \sup_{i\in Z} \sum_{l=-h}^{i} |a_{i-l}| = \sup_{i\in Z} \sum_{j=0}^{i+h} |a_j| = \sum_{j=0}^{\infty} |a_j|,$$

$$\hat{\alpha}_0 = \sup_{i\in Z} \sum_{j=0}^{\infty} |a_{j+i,i}| = \sum_{j=0}^{\infty} |a_j| = \alpha_0,$$

$$S_1=\sup_{i\in Z}\sum_{j=0}^{i-1}\sum_{k=-h}^{j}|\sigma_{jk}^{i}|=\sup_{i\in Z}\sum_{j=0}^{i-1}\sum_{k=-h}^{j}|\sigma_{j-k}^{i-j}|$$

$$=\sup_{i\in Z}\sum_{p=1}^{i}\sum_{k=-h}^{i-p}|\sigma_{i-p-k}^{p}|=\sup_{i\in Z}\sum_{p=1}^{i}\sum_{l=0}^{i-p+h}|\sigma_{l}^{p}|=\sum_{p=1}^{\infty}\sum_{l=0}^{\infty}|\sigma_{l}^{p}|,$$

$$\hat{S}_1=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{l=i}^{j+i-1}|\sigma_{li}^{j+i}|=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{l=i}^{j+i-1}|\sigma_{l-i}^{j+i-l}|$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j-1}|\sigma_{k}^{j-k}|=\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}|\sigma_{k}^{j-k}|=\sum_{k=0}^{\infty}\sum_{p=1}^{\infty}|\sigma_{k}^{p}|=S_1,$$

$$S_0=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{m=i}^{j+i}|\sigma_{mi}^{j+i}|\sum_{l=-h}^{m}|\sigma_{ml}^{j+i}|=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{m=i}^{j+i}|\sigma_{m-i}^{j+i-m}|\sum_{l=-h}^{m}|\sigma_{m-l}^{j+i-m}|$$

$$=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{m=i}^{j+i}|\sigma_{m-i}^{j+i-m}|\sum_{k=0}^{m+h}|\sigma_{k}^{j+i-m}|=\sup_{i\in Z}\sum_{j=0}^{\infty}\sum_{l=0}^{j}|\sigma_{l}^{j-l}|\sum_{k=0}^{l+i+h}|\sigma_{k}^{j-l}|$$

$$=\sum_{l=0}^{\infty}\sum_{j=l}^{\infty}|\sigma_{l}^{j-l}|\sum_{k=0}^{\infty}|\sigma_{k}^{j-l}|=\sum_{p=0}^{\infty}\left(\sum_{k=0}^{\infty}|\sigma_{k}^{p}|\right)^{2}.$$

4.2 Second Way of the Construction of the Lyapunov Functional

1. Represent the right-hand side of (4.1) in the form (1.7) with $\tau=0$,

$$F_1(i,x_i)=\beta_i x_i,\quad \beta_i=\sum_{j=0}^{\infty}a_{j+i,i},$$

$$F_3(i)=F_3(i,x_{-h},\dots,x_i)=-\sum_{l=-h}^{i-1}x_l\sum_{j=i-l}^{\infty}a_{j+l,l},\qquad F_2(i,x_{-h},\dots,x_i)=0,$$

$$G_1(i,j,x_j)=0,\qquad G_2(i,j,x_{-h},\dots,x_j)=\sum_{l=-h}^{j}\sigma_{jl}^{i}x_l,$$

$$j=0,\dots,i,\ i=0,1,\dots.$$

2. The auxiliary equation (1.8) in this case is $y_{i+1}=\beta_i y_i$. Below it is supposed that

$$\sup_{i\in Z}|\beta_i|<1. \tag{4.5}$$

By this condition the function $v_i = y_i^2$ is a Lyapunov function for the auxiliary equation, since $\Delta v_i = (\beta_i^2 - 1)y_i^2$.

3. The functional V_{1i} has to be chosen in the form $V_{1i} = (x_i - F_3(i))^2$.
4. Using (4.2) and representation $x_{i+1} = \beta_i x_i + \Delta F_3(i) + \eta_i$ we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[\big(x_{i+1} - F_3(i+1)\big)^2 - \big(x_i - F_3(i)\big)^2\big]\\ &= \mathbf{E}\big(x_{i+1} - F_3(i+1) - x_i + F_3(i)\big)\big(x_{i+1} - F_3(i+1) + x_i - F_3(i)\big)\\ &= \mathbf{E}\big(\beta_i x_i + \Delta F_3(i) + \eta_i - F_3(i+1) - x_i + F_3(i)\big)\\ &\quad \times \big(\beta_i x_i + \Delta F_3(i) + \eta_i - F_3(i+1) + x_i - F_3(i)\big)\\ &= \mathbf{E}\big((\beta_i - 1)x_i + \eta_i\big)\left((\beta_i + 1)x_i + \eta_i + 2\sum_{l=-h}^{i-1} x_l \sum_{j=i-l}^{\infty} a_{j+l,l}\right)\\ &= \sum_{k=1}^{5} I_k,\end{aligned}$$

where

$$I_1 = \big(\beta_i^2 - 1\big)\mathbf{E}x_i^2, \qquad I_2 = 2\beta_i \mathbf{E}x_i\eta_i, \qquad I_3 = \mathbf{E}\eta_i^2,$$
$$I_4 = 2(1-\beta_i)\mathbf{E}x_i F_3(i), \qquad I_5 = -2\mathbf{E}\eta_i F_3(i).$$

Let us obtain the estimations of the summands $I_2, \ldots, I_5$. Since $\mathbf{E}x_i \sum_{l=-h}^{i} \sigma_{il}^i x_l \xi_{i+1} = 0$, for I_2 we obtain

$$\begin{aligned}|I_2| &\le |\beta_i| \sum_{j=0}^{i-1} \sum_{l=-h}^{j} \big|\sigma_{jl}^i\big|\big(\mathbf{E}x_i^2 + \mathbf{E}x_l^2\big)\\ &= \mathbf{E}x_i^2 |\beta_i| \sum_{j=0}^{i-1} \sum_{l=-h}^{j} \big|\sigma_{jl}^i\big| + |\beta_i| \sum_{k=-h}^{i-1} \sum_{j=k_m}^{i-1} \big|\sigma_{jk}^i\big| \mathbf{E}x_k^2.\end{aligned}$$

For I_3:

$$\begin{aligned}I_3 &= \sum_{j=0}^{i} \mathbf{E}\left(\sum_{l=-h}^{j} \sigma_{jl}^i x_l\right)^2 \le \sum_{j=0}^{i} \sum_{l=-h}^{j} \big|\sigma_{jl}^i\big| \sum_{k=-h}^{j} \big|\sigma_{jk}^i\big| \mathbf{E}x_k^2\\ &= \mathbf{E}x_i^2 \big|\sigma_{ii}^i\big| \sum_{l=-h}^{i} \big|\sigma_{il}^i\big| + \sum_{k=-h}^{i-1} \sum_{j=k_m}^{i} \big|\sigma_{jk}^i\big| \sum_{l=-h}^{j} \big|\sigma_{jl}^i\big| \mathbf{E}x_k^2.\end{aligned}$$

Via (4.5) for I_4 we have

$$|I_4| \le (1-\beta_i)\sum_{l=-h}^{i-1}\left|\sum_{j=i-l}^{\infty} a_{j+l,l}\right|\left(\mathbf{E}x_i^2+\mathbf{E}x_l^2\right)$$
$$=(1-\beta_i)\sum_{l=-h}^{i-1}\left|\sum_{j=i-l}^{\infty} a_{j+l,l}\right|\mathbf{E}x_i^2+(1-\beta_i)\sum_{k=-h}^{i-1}\left|\sum_{j=i-k}^{\infty} a_{j+k,k}\right|\mathbf{E}x_k^2.$$

Since $\mathbf{E}x_l x_k \xi_{j+1}=0$ for $k\le j$, $l\le j$, for I_5 we obtain

$$|I_5| = 2\left|\mathbf{E}\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\sigma_{jl}^i x_l\left(\sum_{k=-h}^{j}x_k\sum_{m=i-k}^{\infty}a_{m+k,k}+\sum_{k=j+1}^{i-1}x_k\sum_{m=i-k}^{\infty}a_{m+k,k}\right)\xi_{j+1}\right|$$
$$=2\left|\mathbf{E}\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\sigma_{jl}^i x_l\sum_{k=j+1}^{i-1}x_k\sum_{m=i-k}^{\infty}a_{m+k,k}\xi_{j+1}\right|$$
$$\le\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{jl}^i\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_{m+k,k}\right|\left(\mathbf{E}x_l^2+\mathbf{E}x_k^2\right)$$
$$=\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{jl}^i\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_{m+k,k}\right|\mathbf{E}x_l^2$$
$$+\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\left|\sigma_{jl}^i\right|\sum_{k=j+1}^{i-1}\left|\sum_{m=i-k}^{\infty}a_{m+k,k}\right|\mathbf{E}x_k^2$$
$$=\sum_{k=-h}^{i-2}\left(\sum_{j=k_m}^{i-2}\left|\sigma_{jk}^i\right|\sum_{l=j+1}^{i-1}\left|\sum_{m=i-l}^{\infty}a_{m+l,l}\right|\right)\mathbf{E}x_k^2$$
$$+\sum_{k=1}^{i-1}\left(\left|\sum_{m=i-k}^{\infty}a_{m+k,k}\right|\sum_{j=0}^{k-1}\sum_{l=-h}^{j}\left|\sigma_{jl}^i\right|\right)\mathbf{E}x_k^2.$$

Thus, as a result

$$\mathbf{E}\Delta V_{1i}\le\mathbf{E}x_i^2\left(\beta_i^2-1+|\beta_i|\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\left|\sigma_{jl}^i\right|+\left|\sigma_{ii}^i\right|\sum_{l=-h}^{i}\left|\sigma_{il}^i\right|\right.$$
$$\left.+(1-\beta_i)\sum_{l=-h}^{i-1}\left|\sum_{j=i-l}^{\infty}a_{j+l,i}\right|\right)+\sum_{k=-h}^{i-1}B_{ik}\mathbf{E}x_k^2,$$

where

$$B_{ik} = |\beta_i| \sum_{j=k_m}^{i-1} |\sigma_{jk}^i| + \sum_{j=k_m}^{i} |\sigma_{jk}^i| \sum_{l=-h}^{j} |\sigma_{jl}^i| + |1-\beta_i| \left| \sum_{j=i-k}^{\infty} a_{j+k,k} \right|$$
$$+ \sum_{j=k_m}^{i-2} |\sigma_{jk}^i| \sum_{m=j+1}^{i-1} \left| \sum_{l=i-m}^{\infty} a_{l+m,m} \right| + \left| \sum_{m=i-k}^{\infty} a_{m+k,k} \right| \sum_{j=0}^{k-1} \sum_{l=-h}^{j} |\sigma_{jl}^i|.$$

So, if conditions (4.5) and

$$\sup_{i\in Z} \Bigg(\beta_i^2 + |\beta_i| \sum_{j=0}^{i-1} \sum_{l=-h}^{j} |\sigma_{jl}^i| + |\sigma_{ii}^i| \sum_{l=-h}^{i} |\sigma_{il}^i|$$
$$+ (1-\beta_i) \sum_{l=-h}^{i-1} \left| \sum_{j=i-l}^{\infty} a_{j+l,l} \right| + \sum_{j=1}^{\infty} B_{j+i,i} \Bigg) < 1 \tag{4.6}$$

hold, then the trivial solution of (4.17) is asymptotically mean square stable.

Example 4.1 Consider the equation

$$x_{i+1} = x_i - b_i x_{i-k} + \sigma_i x_i \xi_{i+1}, \quad k \geq 1. \tag{4.7}$$

The sufficient conditions (4.5) and (4.6) for the asymptotic mean square stability of the trivial solution of (4.7) take the form

$$\inf_{i\in Z} b_i > 0, \qquad \sup_{i\in Z} \left[\sigma_i^2 + b_{i+k} \left(\sum_{l=0}^{2k} b_{i+l} - 2 \right) \right] < 0. \tag{4.8}$$

In particular, in the stationary case ($b_i = b$, $\sigma_i = \sigma$) condition (4.8) takes the form

$$\frac{1 - \sqrt{1-(2k+1)\sigma^2}}{2k+1} < b < \frac{1 + \sqrt{1-(2k+1)\sigma^2}}{2k+1}.$$

4.3 Third Way of the Construction of the Lyapunov Functional

1. Represent the right-hand side of (4.1) in the form (1.7) with $\tau = 1$,

$$F_1(i, x_{i-1}, x_i) = a_{ii} x_i + a_{i,i-1} x_{i-1},$$

$$F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-2} a_{il} x_l, \qquad F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i,j,x_j)=0,$$

$$G_2(i,j,x_{-h},\dots,x_j)=\sum_{l=-h}^{j}\sigma_{jl}^i x_l,\quad j=0,\dots,i,\ i=0,1,\dots.$$

Put

$$x(i)=\begin{pmatrix}x_{i-1}\\ x_i\end{pmatrix},\qquad A(i)=\begin{pmatrix}0 & 1\\ a_{i,i-1} & a_{ii}\end{pmatrix},\qquad B(i)=\begin{pmatrix}0\\ b(i)\end{pmatrix},$$
$$b(i)=\sum_{k=-h}^{i-2}a_{ik}x_k+\sum_{j=0}^{i}\sum_{l=-h}^{j}\sigma_{jl}^i x_l\xi_{j+1}. \tag{4.9}$$

Then (4.1) can be written in matrix form:

$$x(i+1)=A(i)x(i)+B(i). \tag{4.10}$$

2. Put $y(i)=(y_{i-1},y_i)'$ and consider the auxiliary equation

$$y(i+1)=A(i)y(i). \tag{4.11}$$

Let U be a 2×2-matrix with all zero elements except of $u_{22}=1$. The matrix equation

$$A'(i)D(i+1)A(i)-D(i+1)=-U \tag{4.12}$$

has the solution $D(i+1)$ with the elements

$$\begin{aligned}
d_{11}(i+1)&=a_{i,i-1}^2 d_{22}(i+1),\\
d_{12}(i+1)&=\frac{a_{ii}a_{i,i-1}}{1-a_{i,i-1}}d_{22}(i+1),\\
d_{22}(i+1)&=\frac{1-a_{i,i-1}}{(1+a_{i,i-1})[(1-a_{i,i-1})^2-a_{ii}^2]},
\end{aligned} \tag{4.13}$$

which is positive definite if and only if the following conditions hold:

$$\sup_{i\in Z}|a_{i,i-1}|<1,\qquad \sup_{i\in Z}\big(|a_{ii}|+a_{i,i-1}\big)<1. \tag{4.14}$$

Put $v_i=y'(i)D(i)y(i)$, $\Delta D(i)=D(i+1)-D(i)$. Then via (4.11) and (4.12)

$$\begin{aligned}
\Delta v_i&=y'(i+1)D(i+1)y(i+1)-y'(i)D(i)y(i)\\
&=y'(i)\big[A'(i)D(i+1)A(i)-D(i+1)\big]y(i)+y'(i)\Delta D(i)y(i)\\
&\le -y_i^2+\big\|\Delta D(i)\big\|\big(y_i^2+y_{i-1}^2\big),
\end{aligned}$$

where $\|\Delta D(i)\|$ is the operator norm of the matrix $\Delta D(i)$.

Via Theorem 1.2, the function $v_i = y'(i)D(i)y(i)$ is a Lyapunov function for the auxiliary equation (4.11) if the conditions (4.14) hold and $\sup_{i\in Z} \|\Delta D(i)\| < 0.5$.

3. The functional V_{1i} has to be chosen in the form $V_{1i} = x'(i)D(i)x(i)$, where the matrix $D(i)$ is the solution of (4.12).
4. Calculating $\mathbf{E}\Delta V_{1i}$ via (4.9), (4.10), (4.12) and (4.13) we get

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[x'(i+1)D(i+1)x(i+1) - x'(i)D(i)x(i)\big]\\ &= \mathbf{E}\big[\big(A(i)x(i)+B(i)\big)'D(i+1)\big(A(i)x(i)+B(i)\big) - x'(i)D(i)x(i)\big]\\ &= -\mathbf{E}x_i^2 + \mathbf{E}x'(i)\Delta D(i)x(i) + d_{22}(i+1)\mathbf{E}b^2(i)\\ &\quad + 2\mathbf{E}\big(d_{12}(i+1)x_i + d_{22}(i+1)a_{i,i-1}x_{i-1} + d_{22}(i+1)a_{ii}x_i\big)b(i)\\ &\le -\mathbf{E}x_i^2 + \big\|\Delta D(i)\big\|\big(\mathbf{E}x_i^2 + \mathbf{E}x_{i-1}^2\big) + \sum_{j=0}^{7} I_j,\end{aligned}$$

where

$$I_1 = d_{22}(i+1)\mathbf{E}\left(\sum_{l=-h}^{i-2} a_{il}x_l\right)^2,$$

$$I_2 = 2d_{22}(i+1)\mathbf{E}\sum_{k=-h}^{i-2} a_{ik}x_k \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1},$$

$$I_3 = d_{22}(i+1)\mathbf{E}\left(\sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1}\right)^2,$$

$$I_4 = 2d_{22}(i+1)\frac{a_{ii}}{1-a_{i,i-1}}\mathbf{E}x_i \sum_{l=-h}^{i-2} a_{il}x_l,$$

$$I_5 = 2d_{22}(i+1)\frac{a_{ii}}{1-a_{i,i-1}}\mathbf{E}x_i \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1},$$

$$I_6 = 2d_{22}(i+1)a_{i,i-1}\mathbf{E}x_{i-1} \sum_{l=-h}^{i-2} a_{il}x_l,$$

$$I_7 = 2d_{22}(i+1)a_{i,i-1}\mathbf{E}x_{i-1} \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1}.$$

Let us estimate these seven summands. Put

$$\alpha_2 = \sup_{i\in Z} \sum_{l=-h}^{i-2} |a_{il}|, \qquad S_k = \sup_{i\in Z} \sum_{j=0}^{i-k} \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right|, \quad k = 1, 2, \ldots .$$

For I_1 we have

$$I_1 \le d_{22}(i+1) \sum_{l=-h}^{i-2} |a_{il}| \sum_{k=-h}^{i-2} |a_{ik}| \mathbf{E} x_k^2 \le d_{22}(i+1)\alpha_2 \sum_{k=-h}^{i-2} |a_{ik}| \mathbf{E} x_k^2 .$$

Since $\mathbf{E} x_k x_l \xi_{j+1} = 0$ for $k, l \le j$ and $\mathbf{E} x_l^2 \xi_{j+1}^2 = \mathbf{E} x_l^2$ for $l \le j$, we have

$$\begin{aligned}
|I_2| &= 2d_{22}(i+1)\left|\mathbf{E} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1} \left(\sum_{k=-h}^{j} a_{ik} x_k + \sum_{k=j+1}^{i-2} a_{ik} x_k \right)\right| \\
&= 2d_{22}(i+1)\left|\mathbf{E} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \sigma_{jl}^i x_l \xi_{j+1} \left(\sum_{k=j+1}^{i-2} a_{ik} x_k \right)\right| \\
&\le d_{22}(i+1) \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \sum_{k=j+1}^{i-2} |a_{ik}| \left(\mathbf{E} x_k^2 + \mathbf{E} x_l^2\right) \\
&= d_{22}(i+1)\left(\sum_{k=-h}^{i-3} \sum_{j=k_m}^{i-3} \left|\sigma_{jk}^i\right| \sum_{l=j+1}^{i-2} |a_{il}| \mathbf{E} x_k^2 + \sum_{k=1}^{i-2} |a_{ik}| \sum_{j=0}^{k-1} \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \mathbf{E} x_k^2 \right) \\
&\le d_{22}(i+1)\left(\alpha_2 \sum_{k=-h}^{i-3} \sum_{j=k_m}^{i-3} \left|\sigma_{jk}^i\right| \mathbf{E} x_k^2 + S_3 \sum_{k=1}^{i-2} |a_{ik}| \mathbf{E} x_k^2 \right)
\end{aligned}$$

and for I_3 we obtain

$$\begin{aligned}
I_3 &= d_{22}(i+1) \sum_{j=0}^{i} \mathbf{E} \left(\sum_{l=-h}^{j} \sigma_{jl}^i x_l \right)^2 \le d_{22}(i+1) \sum_{j=0}^{i} \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \sum_{k=-h}^{j} \left|\sigma_{jk}^i\right| \mathbf{E} x_k^2 \\
&= d_{22}(i+1) \sum_{k=-h}^{i} \sum_{j=k_m}^{i} \left|\sigma_{jk}^i\right| \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \mathbf{E} x_k^2 = d_{22}(i+1)\Bigg(\left|\sigma_{ii}^i\right| \sum_{l=-h}^{i} \left|\sigma_{il}^i\right| \mathbf{E} x_i^2 \\
&\quad + \sum_{j=i-1}^{i} \left|\sigma_{j,i-1}^i\right| \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \mathbf{E} x_{i-1}^2 + \sum_{k=-h}^{i-2} \sum_{j=k_m}^{i} \left|\sigma_{jk}^i\right| \sum_{l=-h}^{j} \left|\sigma_{jl}^i\right| \mathbf{E} x_k^2 \Bigg).
\end{aligned}$$

Similar for I_4,

$$|I_4| \le d_{22}(i+1)\frac{|a_{ii}|}{1-a_{i,i-1}}\sum_{l=-h}^{i-2}|a_{il}|\big(\mathbf{E}x_l^2+\mathbf{E}x_i^2\big)$$

$$= d_{22}(i+1)\frac{|a_{ii}|}{1-a_{i,i-1}}\left(\alpha_2\mathbf{E}x_i^2+\sum_{k=-h}^{i-2}|a_{ik}|\mathbf{E}x_k^2\right)$$

and for I_5,

$$|I_5| \le d_{22}(i+1)\frac{|a_{ii}|}{1-a_{i,i-1}}\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\big|\sigma_{jl}^i\big|\big(\mathbf{E}x_l^2+\mathbf{E}x_i^2\big)$$

$$\le d_{22}(i+1)\frac{|a_{ii}|}{1-a_{i,i-1}}\left(S_1\mathbf{E}x_i^2+\big|\sigma_{i-1,i-1}^i\big|\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}\sum_{j=k_m}^{i-1}\big|\sigma_{jk}^i\big|\mathbf{E}x_k^2\right).$$

At last for I_6,

$$|I_6| \le d_{22}(i+1)|a_{i,i-1}|\sum_{l=-h}^{i-2}|a_{il}|\big(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2\big)$$

$$\le d_{22}(i+1)|a_{i,i-1}|\left(\alpha_2\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}|a_{ik}|\mathbf{E}x_k^2\right)$$

and for I_7,

$$|I_7| \le d_{22}(i+1)|a_{i,i-1}|\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\big|\sigma_{jl}^i\big|\big(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2\big)$$

$$= d_{22}(i+1)|a_{i,i-1}|\left(S_2\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}\sum_{j=k_m}^{i-2}\big|\sigma_{jk}^i\big|\mathbf{E}x_k^2\right).$$

As a result we get

$$\mathbf{E}\Delta V_{1i} \le \big(-1+\gamma_0(i)\big)\mathbf{E}x_i^2+\gamma_1(i)\mathbf{E}x_{i-1}^2+\sum_{k=-h}^{i-2}P_{ik}\mathbf{E}x_k^2,$$

where

$$\gamma_0(i)=\big\|\Delta D(i)\big\|+d_{22}(i+1)\left[\big|\sigma_{ii}^i\big|\sum_{l=-h}^{i}\big|\sigma_{il}^i\big|+\frac{|a_{ii}|}{1-a_{i,i-1}}(\alpha_2+S_1)\right],$$

$$\gamma_1(i) = \|\Delta D(i)\| + d_{22}(i+1)\Bigg[\sum_{j=i-1}^{i} |\sigma^i_{j,i-1}| \sum_{l=-h}^{j} |\sigma^i_{jl}| + |a_{ii}| \frac{|\sigma^i_{i-1,i-1}|}{1 - a_{i,i-1}} + |a_{i,i-1}|(\alpha_2 + S_2)\Bigg]$$

and

$$P_{ik} = d_{22}(i+1)\Bigg[(\alpha_2 + S_3)|a_{ik}| + \alpha_2 \sum_{j=k_m}^{i-3} |\sigma^i_{jk}| + \sum_{j=k_m}^{i} |\sigma^i_{jk}| \sum_{l=-h}^{j} |\sigma^i_{jl}| + \frac{|a_{ii}|}{1 - a_{i,i-1}}\Bigg(|a_{ik}| + \sum_{j=k_m}^{i-1} |\sigma^i_{jk}|\Bigg) + |a_{i,i-1}|\Bigg(|a_{ik}| + \sum_{j=k_m}^{i-2} |\sigma^i_{jk}|\Bigg)\Bigg]$$

$$k := -h, -h+1, \dots, i-2.$$

Via Theorem 1.2 inequalities (4.14) and

$$\sup_{i \in Z}\Bigg(\gamma_0(i) + \gamma_1(i) + \sum_{j=2}^{\infty} P_{j+i-1,i-1}\Bigg) < 1 \tag{4.15}$$

are sufficient conditions for the asymptotic mean square stability of the trivial solution of (4.1).

Example 4.2 Suppose that the coefficients of the difference equation

$$x_{i+1} = a_{0i}x_i + a_{1i}x_{i-1} + \sigma_i x_i \xi_{i+1} \tag{4.16}$$

have small enough fluctuations and satisfy the conditions

$$a_{ki} = a_k + \epsilon_{ki}, \quad |\epsilon_{ki}| \le \epsilon, \ k = 0, 1, \qquad \sigma_i^2 \le \sigma^2, \quad i \in Z.$$

For (4.16) $\gamma_0(i) = \|\Delta D(i)\| + d_{22}(i+1)\sigma_i^2$, $\gamma_1(i) = \|\Delta D(i)\|$, $P_{ik} = 0$. So, condition (4.15) takes the form

$$\sup_{i \in Z}\big(2\|\Delta D(i)\| + d_{22}(i+1)\sigma_i^2\big) < 1.$$

It is clear that if the stability conditions (4.14) and (4.15) hold for $\epsilon = 0$ then these conditions hold for small enough $\epsilon > 0$ too. In fact, the norm $\|\Delta D(i)\|$ satisfies the estimation $\|\Delta D(i)\| \le C\epsilon$ for some positive C that depends on a_0, a_1 only. So, if

$$|a_1| < 1 - \epsilon, \quad |a_0| < 1 - a_1 - 2\epsilon,$$

$$2\|\Delta D(i)\| + d_{22}(i+1)\sigma_i^2$$
$$\leq 2C\epsilon + \frac{\sigma^2(1-a_1+\epsilon)}{(1+a_1-\epsilon)[(1-a_1-2\epsilon)^2 - a_0^2]} < 1$$

then the trivial solution of (4.16) is asymptotically mean square stable.

Note that in the stationary case ($a_{0i} = a_0$, $a_{1i} = a_1$, $\sigma_i = \sigma$) these conditions coincide with (2.15) and (2.16).

4.4 Systems with Monotone Coefficients

Consider the linear equation

$$x_{i+1} = -\sum_{j=0}^{i} a_{ij} x_j + \sum_{j=0}^{i} \sigma_{ij} x_j \xi_{i+1}, \tag{4.17}$$

where x_0 is a $\mathfrak{F}_0$-adapted random variable, $\mathbf{E}x_0^2 < \infty$.

Equation (4.17) is a particular case of (4.1) and stability conditions (4.4) or (4.5) and (4.6) obtained for (4.1) can be applied to (4.17) too. But these stability conditions contain some assumptions about convergence of series from coefficients a_{ij}. These assumptions sometimes are very limiting. Below, the stability conditions for (4.17) are obtained without any assumptions about convergence of the series from coefficients a_{ij}. These stability conditions are obtained by virtue of the construction of special Lyapunov functionals and are formulated in terms of fixed sign and monotone sequences of coefficients.

Following the procedure of the construction of Lyapunov functionals consider the auxiliary equation for (4.17) in the form $y_{i+1} = 0$, $i \in Z$. The function $v_i = y_i^2$ is a Lyapunov function for this equation, since $\Delta v_i = y_{i+1}^2 - y_i^2 = -y_i^2$.

The Lyapunov functional for (4.17) is constructed in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$ and

$$V_{2i} = \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} x_k \right)^2.$$

It is supposed that the numbers α_{ij}, which will be defined below, satisfy the inequalities

$$0 \leq \alpha_{i+1,j} \leq \alpha_{ij}, \quad i \in Z, \ j = 0, 1, \dots, i+1. \tag{4.18}$$

Note that

$$\Delta V_{1i} = -x_i^2 + x_{i+1}^2. \tag{4.19}$$

Calculating ΔV_{2i}, we obtain

$$\begin{aligned}\Delta V_{2i} &= \sum_{j=0}^{i+1}\alpha_{i+1,j}\left(\sum_{k=j}^{i+1}x_k\right)^2 - \sum_{j=0}^{i}\alpha_{ij}\left(\sum_{k=j}^{i}x_k\right)^2 \\ &= \sum_{j=0}^{i+1}(\alpha_{i+1,j}-\alpha_{ij})\left(\sum_{k=j}^{i+1}x_k\right)^2 + \sum_{j=0}^{i}\alpha_{ij}\left[\left(\sum_{k=j}^{i+1}x_k\right)^2 - \left(\sum_{k=j}^{i}x_k\right)^2\right] \\ &\quad + \alpha_{i,i+1}x_{i+1}^2.\end{aligned}$$

Setting

$$\gamma_{ij} = \sum_{k=0}^{j}\alpha_{ik}, \quad j = 0, 1, \ldots, i+1, \tag{4.20}$$

and using (4.18), we get

$$\begin{aligned}\Delta V_{2i} &\le \sum_{j=0}^{i}\alpha_{ij}\left(x_{i+1}^2 + 2x_{i+1}\sum_{k=j}^{i}x_k\right) + \alpha_{i,i+1}x_{i+1}^2 \\ &= \gamma_{i,i+1}x_{i+1}^2 + 2x_{i+1}\sum_{j=0}^{i}\gamma_{ij}x_j. \end{aligned}\tag{4.21}$$

As a result for the functional $V_i = V_{1i} + V_{2i}$ by virtue of (4.17), (4.19) and (4.21) we have

$$\begin{aligned}\Delta V_i &\le -x_i^2 + x_{i+1}\left[2\sum_{j=0}^{i}\gamma_{ij}x_j + (1+\gamma_{i,i+1})x_{i+1}\right] \\ &= -x_i^2 + x_{i+1}\left[2\sum_{j=0}^{i}\gamma_{ij}x_j + (1+\gamma_{i,i+1})\left(-\sum_{j=0}^{i}a_{ij}x_j + \sum_{j=0}^{i}\sigma_{ij}x_j\xi_{i+1}\right)\right] \\ &= -x_i^2 + x_{i+1}\sum_{j=0}^{i}\big[2\gamma_{ij} - (1+\gamma_{i,i+1})a_{ij}\big]x_j + (1+\gamma_{i,i+1})\sum_{j=0}^{i}\sigma_{ij}x_j\xi_{i+1}x_{i+1}.\end{aligned}$$

Suppose that

$$2\gamma_{ij} = (1+\gamma_{i,i+1})a_{ij}, \quad j = 0, 1, \ldots, i. \tag{4.22}$$

Via (4.17)

$$\begin{aligned}\mathbf{E}\sum_{j=0}^{i}\sigma_{ij}x_j\xi_{i+1}x_{i+1} &= \mathbf{E}\sum_{j=0}^{i}\sigma_{ij}x_j\xi_{i+1}\left(-\sum_{j=0}^{i}a_{ij}x_j + \sum_{j=0}^{i}\sigma_{ij}x_j\xi_{i+1}\right) \\ &= \mathbf{E}\left(\sum_{j=0}^{i}\sigma_{ij}x_j\right)^2 \le \sum_{j=0}^{i}B_{ij}\mathbf{E}x_j^2,\end{aligned}$$

where

$$B_{ij} = |\sigma_{ij}| \sum_{k=0}^{i} |\sigma_{ik}|. \tag{4.23}$$

Put $\gamma = \sup_{i\in Z} \gamma_{i,i+1}$. As a result

$$\mathbf{E}\Delta V_i \le -\mathbf{E}x_i^2 + (1+\gamma) \sum_{j=0}^{i} B_{ij} \mathbf{E}x_j^2. \tag{4.24}$$

Via (4.23)

$$\sigma^2 = \sup_{i\in Z} \sum_{j=0}^{\infty} B_{j+i,i} = \sup_{i\in Z} \sum_{j=0}^{\infty} |\sigma_{j+i,i}| \sum_{k=0}^{j+i} |\sigma_{j+i,k}|. \tag{4.25}$$

From Theorem 1.2 and (4.25) it follows that the inequality

$$(1+\gamma)\sigma^2 < 1 \tag{4.26}$$

is a sufficient condition for the asymptotic mean square stability of the trivial solution of (4.17).

In order to get the stability condition in terms of the parameters of (4.17), transform inequality (4.26) in the following way. Let $a_{i,i+1}$, $i \in Z$, be a monotone nondecreasing sequence such that $\alpha = \sup_{i\in Z} a_{i,i+1} < 2$. Then from (4.22) for $j = i+1$ it follows that

$$\gamma_{i,i+1} = \frac{a_{i,i+1}}{2 - a_{i,i+1}}, \qquad \gamma = \frac{\alpha}{2-\alpha}. \tag{4.27}$$

By virtue of (4.27), the condition (4.26) takes the form

$$\alpha < 2\bigl(1 - \sigma^2\bigr). \tag{4.28}$$

Substituting (4.27) into (4.22) we obtain

$$\gamma_{ij} = \frac{a_{ij}}{2 - a_{i,i+1}}, \quad j = 0, 1, \dots, i+1.$$

From this and (4.20) and (4.18) it follows that

$$\begin{aligned} \alpha_{ij} &= \gamma_{ij} - \gamma_{i,j-1} = \frac{a_{ij} - a_{i,j-1}}{2 - a_{i,i+1}} \ge 0, \quad j = 1, \dots, i+1, \\ \alpha_{i0} &= \gamma_{i0} = \frac{a_{i0}}{2 - a_{i,i+1}} \ge 0. \end{aligned} \tag{4.29}$$

Via (4.18), (4.28) and (4.29) we have

$$\begin{aligned}
&a_{ij} \geq a_{i,j-1} \geq a_{i0} \geq 0, \\
&a_{i+1,j-1} - a_{i+1,j} - a_{i,j-1} + a_{ij} \geq 0, \\
&\sup_{i \in Z}(a_{i+1,i+1} + a_{ii} - a_{i+1,i}) < 2\big(1 - \sigma^2\big).
\end{aligned} \tag{4.30}$$

So, if the parameters of (4.17) satisfy the inequalities (4.30) then the trivial solution of (4.17) is asymptotically mean square stable.

Remark 4.2 In the stationary case, i.e. $a_{ij} = a_{i-j}$, $\sigma_{ij} = \sigma_{i-j}$, conditions (4.30) take the form

$$\begin{aligned}
&a_i \geq a_{i+1} \geq 0, \quad i = 0, 1, \ldots, \\
&a_{i+2} - 2a_{i+1} + a_i \geq 0, \\
&2a_0 - a_1 < 2\big(1 - \sigma^2\big),
\end{aligned} \tag{4.31}$$

where via (4.25)

$$\sigma^2 = \sup_{i \in Z} \sum_{j=0}^{\infty} |\sigma_j| \sum_{k=0}^{j+i} |\sigma_{j+i-k}| = \sum_{j=0}^{\infty} |\sigma_j| \sup_{i \in Z} \sum_{k=0}^{j+i} |\sigma_k| = \left(\sum_{j=0}^{\infty} |\sigma_j| \right)^2.$$

Example 4.3 Consider the equation

$$x_{i+1} = -ax_i - b \sum_{j=1}^{i} x_{i-j} + \sigma x_{i-1} \xi_{i+1}. \tag{4.32}$$

From (4.31) we obtain sufficient conditions for the asymptotic mean square stability of the trivial solution of (4.32) in the form

$$0 \leq b \leq a < \frac{b}{2} + 1 - \sigma^2. \tag{4.33}$$

Consider the particular case $a = b$. In this case (4.32) takes the form

$$x_{i+1} = -bS_i + \sigma x_{i-1} \xi_{i+1}, \quad S_i = \sum_{j=0}^{i} x_j, \tag{4.34}$$

and for the functional

$$V_i = x_i^2 + \gamma S_i^2, \quad \gamma = \frac{b}{2 - b},$$

we obtain the exact equality

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big[-x_i^2 + x_{i+1}^2 + \gamma\big(S_{i+1}^2 - S_i^2\big)\big] \\
&= \mathbf{E}\big[-x_i^2 + x_{i+1}^2 + \gamma x_{i+1}(x_{i+1} + 2S_i)\big] \\
&= \mathbf{E}\big[-x_i^2 + (1+\gamma)x_{i+1}^2 + 2\gamma x_{i+1}S_i\big] \\
&= \mathbf{E}\big[-x_i^2 + (1+\gamma)(-bS_i + \sigma x_{i-1}\xi_{i+1})^2 + 2\gamma(-bS_i + \sigma x_{i-1}\xi_{i+1})S_i\big)\big] \\
&= \mathbf{E}\big[-x_i^2 + (1+\gamma)\big(b^2S_i^2 + \sigma^2 x_{i-1}^2\big) - 2\gamma bS_i^2\big] \\
&= \mathbf{E}\big[-x_i^2 + (1+\gamma)\sigma^2 x_{i-1}^2 + \big((1+\gamma)b - 2\gamma\big)bS_i^2\big] \\
&= -\mathbf{E}x_i^2 + \frac{2\sigma^2}{2-b}\mathbf{E}x_{i-1}^2.
\end{aligned}$$

Via Corollary 1.2, the condition $2\sigma^2 < 2-b$, which is a particular case of condition (4.33) if $a=b$, and which can also be written in the form

$$0 \le b < 2\big(1-\sigma^2\big), \tag{4.35}$$

is the necessary and sufficient condition for the asymptotic mean square stability of the trivial solution of (4.34).

Example 4.4 Consider the equation

$$x_{i+1} = -ax_i - \sum_{j=1}^{i} b^j x_{i-j} + \sigma x_{i-1}\xi_{i+1}. \tag{4.36}$$

From (4.31) it follows that sufficient conditions for the asymptotic mean square stability of the trivial solution of (4.36) are

$$0 \le b \le 1, \quad 2b - b^2 \le a < \frac{b}{2} + 1 - \sigma^2. \tag{4.37}$$

Chapter 5
Some Peculiarities of the Method

In this chapter some peculiarities of the proposed method of Lyapunov functionals construction are considered.

5.1 Necessary and Sufficient Condition

Here the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the stochastic linear difference equation is considered in more detail.

Consider the scalar difference equation

$$\begin{aligned} x_{i+1} &= \sum_{j=0}^{k} a_j x_{i-j} + \sigma x_{i-m}\xi_{i+1}, \quad i \in Z, \\ x_i &= \varphi_i, \quad i \in Z_0, \qquad h = \max(k, m). \end{aligned} \tag{5.1}$$

Suppose that the matrix equation

$$A'DA - D = -U, \tag{5.2}$$

where the square matrix $U = \|u_{ij}\|$ of dimension $k+1$ has all elements zero except for $u_{k+1,k+1} = 1$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_k & a_{k-1} & a_{k-2} & \dots & a_0 \end{pmatrix}, \tag{5.3}$$

has a positive semidefinite solution D with $d_{k+1,k+1} > 0$. Via Remarks 3.1, and 3.2 the inequality

$$\sigma^2 < d_{k+1,k+1}^{-1} \tag{5.4}$$

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_5, © Springer-Verlag London Limited 2011

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.1).

Note that for each $k = 0, 1, \ldots$ the matrix equation (5.2) is a system of $(k + 1)(k + 2)/2$ equations. Consider the different particular cases of (5.2) and condition (5.4).

1. Let $k = 0$, $a_0 = a$. In this case (5.1) and (5.2) have, respectively, the forms

$$x_{i+1} = ax_i + \sigma x_{i-m}\xi_{i+1}, \tag{5.5}$$

   and

$$d_{11}\big(a^2 - 1\big) = -1.$$

   The necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.5) is

$$a^2 + \sigma^2 < 1.$$

2. Let $k = 1$, $a_0 = a$, $a_1 = b$. In this case (5.1) has the form

$$x_{i+1} = ax_i + bx_{i-1} + \sigma x_{i-m}\xi_{i+1}, \tag{5.6}$$

   and the solution of matrix (5.2) is obtained in (3.11). In particular,

$$d_{22} = \frac{1 - b}{(1 + b)[(1 - b)^2 - a^2]}.$$

   So, the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.6) takes the form

$$|b| < 1, \ |a| < 1 - b, \quad \sigma^2 < 1 - b^2 - a^2\frac{1 + b}{1 - b}. \tag{5.7}$$

   This stability condition coincides with (2.15) and (2.16). Corresponding stability regions are shown in Fig. 2.5 for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.
3. Let $k = 2$. In this case (5.1) has the form

$$x_{i+1} = a_0x_i + a_1x_{i-1} + a_2x_{i-2} + \sigma x_{i-m}\xi_{i+1}, \tag{5.8}$$

   and matrix (5.2) is equivalent to the system of (3.17) with solution (3.18), the condition (5.4) takes the form

$$\sigma^2 < 1 - a_0^2 - a_1^2 - a_2^2 - 2a_0a_1a_2 - \frac{2(a_0 + a_2)(a_0 + a_1a_2)(a_1 + a_0a_2)}{1 - a_1 - a_2(a_0 + a_2)}. \tag{5.9}$$

   If, in particular, $a_0 = a$, $a_1 = 0$, $a_2 = b$, then (5.8) and condition (5.9), respectively, are

$$x_{i+1} = ax_i + bx_{i-2} + \sigma x_{i-m}\xi_{i+1} \tag{5.10}$$

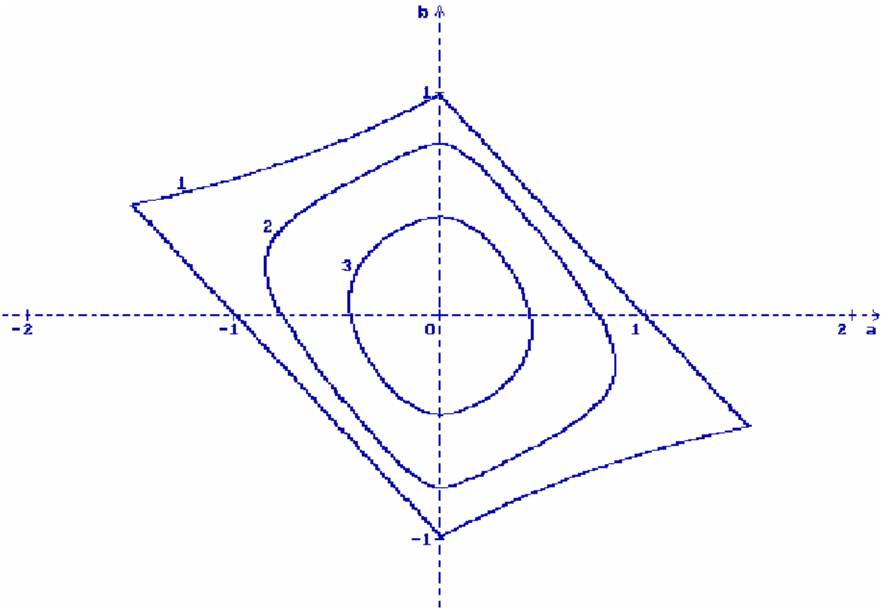


Fig. 5.1 Stability regions for (5.10): (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

and

$$\sigma^2 < 1 - b^2 - a^2 \frac{1 + b(a+b)}{1 - b(a+b)}.$$

In Fig. 5.1 the corresponding regions of asymptotic mean square stability of the trivial solution of (5.10) are shown for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

If $a_0 = 0$, $a_1 = a$, $a_2 = b$, then (5.8) and condition (5.9), respectively, are

$$x_{i+1} = ax_{i-1} + bx_{i-2} + \sigma x_{i-m}\xi_{i+1} \tag{5.11}$$

and

$$\sigma^2 < 1 - b^2 - a^2 \frac{1 - a + b^2}{1 - a - b^2}.$$

In Fig. 5.2 the corresponding regions of asymptotic mean square stability of the trivial solution of (5.11) are shown for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

If in (5.8) $a_0 = a$, $a_1 = b$, $a_2 = b^2$ then (5.8) and condition (5.9), respectively, are

$$x_{i+1} = ax_i + bx_{i-1} + b^2 x_{i-2} + \sigma x_{i-m}\xi_{i+1} \tag{5.12}$$

and

$$\sigma^2 < 1 - a^2 - b^2 - b^4 - 2ab^3 - \frac{2b(1+ab)(a+b^2)(a+b^3)}{1 - b - b^4 - ab^2}.$$

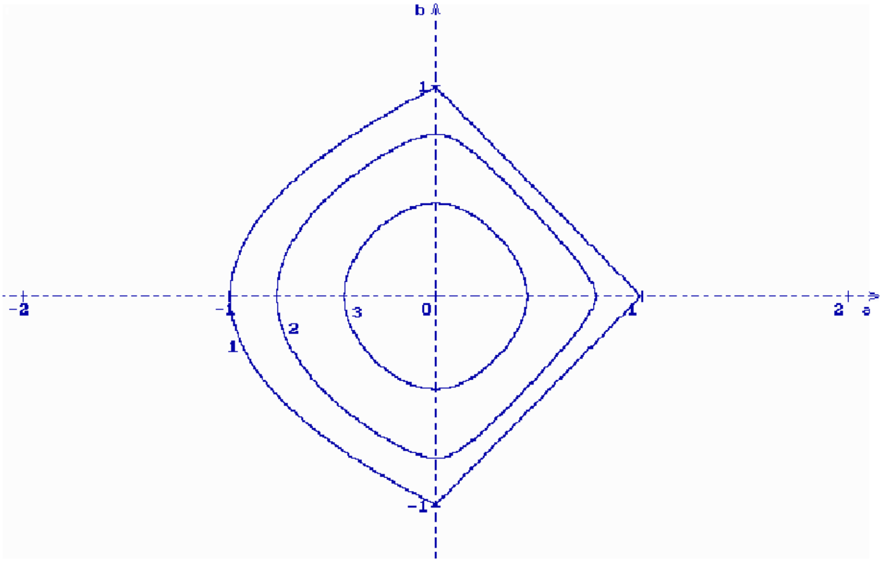


Fig. 5.2 Stability regions for (5.11): (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

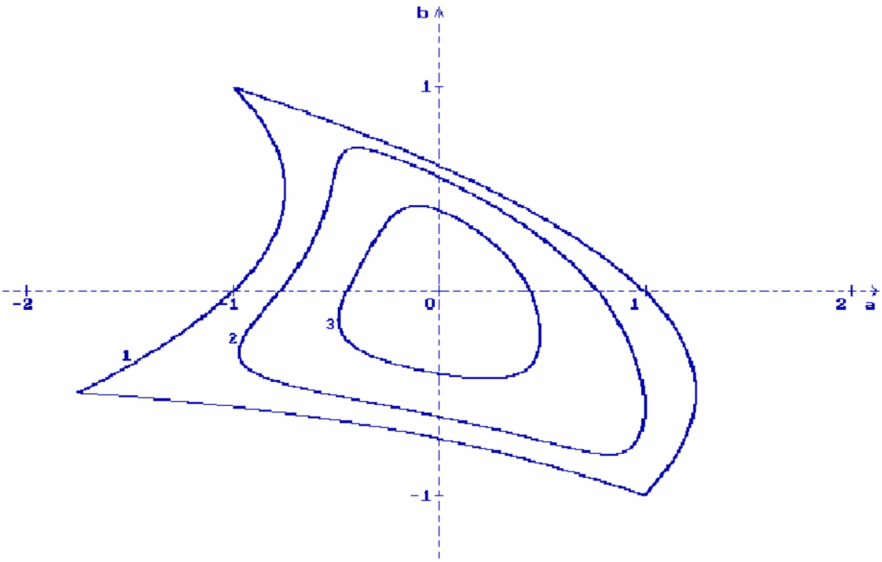


Fig. 5.3 Stability regions for (5.12): (*1*) $\sigma^2 = 0$, (*2*) $\sigma^2 = 0.4$, (*3*) $\sigma^2 = 0.8$

In Fig. 5.3 the corresponding regions of asymptotic mean square stability of the trivial solution of (5.12) are shown for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

Let us consider (5.1) again but suppose that $x_i \in \mathbf{R}^n$, a_j, $j = 0, \ldots, k$, and σ are n-dimension square matrices. Consider also the vector $x(i) =$

$(x_{i-k}, \dots, x_{i-1}, x_i)' \in \mathbf{R}^{n(k+1)}$, the matrix $B = (0, \dots, 0, \sigma')'$ of $n(k+1) \times n$-dimension and two square matrices A and U of $n(k+1)$-dimension such that

$$A = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \\ a_k & a_{k-1} & a_{k-2} & \dots & a_0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & P \end{pmatrix},$$

where I is the n-dimension identity matrix and P is an arbitrary positive definite n-dimension matrix. Then (5.1) can be represented in the form

$$x(i+1) = Ax(i) + Bx_{i-l}\xi_{i+1}. \tag{5.13}$$

Theorem 5.1 *Let for some positive definite matrix P the matrix equation*

$$A'DA - D = -U \tag{5.14}$$

have a positive semidefinite solution

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \dots & D_{1k} & D_{1,k+1} \\ D_{21} & D_{22} & D_{23} & \dots & D_{2k} & D_{2,k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ D_{k1} & D_{k2} & D_{k3} & \dots & D_{kk} & D_{k,k+1} \\ D_{k+1,1} & D_{k+1,2} & D_{k+1,3} & \dots & D_{k+1,k} & D_{k+1,k+1} \end{pmatrix},$$

where D_{ij}, $i, j = 1, \dots, k+1$, are square n-dimension matrices, such that $D'_{ij} = D_{ji}$ and $D_{k+1,k+1}$ is an n-dimension positive definite matrix. Then the trivial solution of (5.1) is asymptotically mean square stable if and only if

$$\sigma' D_{k+1,k+1}\sigma < P, \tag{5.15}$$

i.e., the matrix $\sigma' D_{k+1,k+1}\sigma - P$ is a negative definite one.

To prove the theorem it is enough to show that the functional

$$V_i = x'(i)Dx(i) + \sum_{j=1}^{l} x'_{i-j} b' D_{k+1,k+1} b x_{i-j},$$

where D and $D_{k+1,k+1}$ are defined by matrix (5.14), satisfies the condition

$$\mathbf{E}\Delta V_i = \mathbf{E}x'_i(\sigma' D_{k+1,k+1}\sigma - P)x_i.$$

Example 5.1 Consider the system of the stochastic difference equations

$$\begin{aligned} y_{i+1} &= \alpha z_i + \beta y_{i-1} + \sigma_0 y_{i-l}\xi_{i+1}, \\ z_{i+1} &= \alpha y_i + \beta z_{i-1} + \sigma_0 z_{i-l}\xi_{i+1}. \end{aligned} \tag{5.16}$$

Let us show that the inequalities

$$-1 < \beta < 1 - |\alpha| \tag{5.17}$$

and

$$\sigma_0^2 + \beta^2 + \alpha^2 \frac{1+\beta}{1-\beta} < 1 \tag{5.18}$$

are the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of system (5.16).

Putting

$$x_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}, \qquad a_0 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

let us represent the system (5.16) in the form (5.1) as follows:

$$x_{i+1} = a_0 x_i + a_1 x_{i-1} + b x_{i-l} \xi_{i+1}.$$

The matrix (5.14) with the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & \alpha \\ 0 & \beta & \alpha & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_{11} & p_{12} \\ 0 & 0 & p_{12} & p_{22} \end{pmatrix},$$

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{12}' & D_{22} \end{pmatrix}, \quad D_{11} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}, \ D_{12} = \begin{pmatrix} d_{13} & d_{14} \\ d_{23} & d_{24} \end{pmatrix},$$

$$D_{22} = \begin{pmatrix} d_{33} & d_{34} \\ d_{34} & d_{44} \end{pmatrix},$$

takes the form of the system of the ten equations

$$\begin{aligned}
&\beta^2 d_{33} = d_{11}, \qquad \beta^2 d_{34} = d_{12}, \qquad \beta^2 d_{44} = d_{22}, \\
&\beta d_{13} + \alpha\beta d_{34} = d_{13}, \qquad \beta d_{23} + \alpha\beta d_{33} = d_{14}, \\
&\beta d_{14} + \alpha\beta d_{44} = d_{23}, \qquad \beta d_{24} + \alpha\beta d_{34} = d_{24}, \\
&d_{11} + 2\alpha d_{14} + \alpha^2 d_{44} = d_{33} - p_{11}, \\
&d_{12} + \alpha(d_{13} + d_{24}) + \alpha^2 d_{34} = d_{34} - p_{12}, \\
&d_{22} + 2\alpha d_{23} + \alpha^2 d_{33} = d_{44} - p_{22}.
\end{aligned} \tag{5.19}$$

Put now

$$\gamma = \frac{\alpha\beta}{1-\beta}, \qquad \delta = \left(1-\beta^2-\alpha^2\frac{1+\beta}{1-\beta}\right)^{-1},$$
$$\mu = 1-\beta^2-\frac{2\alpha^2\beta}{1-\beta^2}, \qquad \nu = \alpha^2\frac{1+\beta^2}{1-\beta^2}.$$

Then the solution of system (5.19) can be represented as follows:

$$\begin{aligned}
&d_{11} = \beta^2 d_{33}, \qquad d_{12} = \beta^2 d_{34}, \qquad d_{22} = \beta^2 d_{44}, \qquad d_{13} = d_{24} = \gamma d_{34},\\
&d_{14} = \frac{\alpha\beta}{1-\beta^2}(d_{33}+\beta d_{44}), \qquad d_{23} = \frac{\alpha\beta}{1-\beta^2}(\beta d_{33}+d_{44}),\\
&d_{34} = \delta p_{12}, \qquad d_{33} = \frac{\mu p_{11}+\nu p_{22}}{\mu^2-\nu^2}, \qquad d_{44} = \frac{\mu p_{22}+\nu p_{11}}{\mu^2-\nu^2}.
\end{aligned} \tag{5.20}$$

It is easy to check that by conditions (5.17) and (5.18) the matrix D with elements (5.20) is a positive semidefinite matrix and the matrix D_{22} is a positive definite matrix.

Choosing, for example, the matrix P with the elements $p_{11} = p_{22} = p > 0$, $p_{12} = 0$ we obtain

$$D = p\delta\begin{pmatrix} \beta^2 & 0 & 0 & \gamma \\ 0 & \beta^2 & \gamma & 0 \\ 0 & \gamma & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\sigma' D_{22}\sigma = p\delta\sigma_0^2 I$. From (5.15) it follows that the necessary and sufficient condition for the asymptotic mean square stability of the trivial solution of the system (5.16) is $\delta\sigma_0^2 < 1$, which is equivalent to (5.18).

5.2 Different Ways of Estimation

In the previous items it was shown that via different representations of type (1.7) of the initial equation one can construct different Lyapunov functionals and therefore get different stability conditions.

Here it will be shown that using different ways of estimation of $\mathbf{E}\Delta V_{1i}$ one can also construct different Lyapunov functionals and as a result obtain different stability conditions.

Consider the equation

$$x_{i+1} = ax_i + b\sum_{j=1}^{k}(k+1-j)x_{i-j} + \sigma\sum_{j=0}^{m}(m+1-j)x_{i-j}\xi_{i+1}. \tag{5.21}$$

This equation is a particular case of (3.1) for

$$a_0 = a, \qquad a_j = b(k+1-j), \quad j = 1, \dots, k, \qquad a_j = 0, \quad j > k \geq 1,$$
$$\sigma_j^l = 0, \quad l > 0, \; j \geq 0,$$
$$\sigma_j^0 = \sigma(m+1-j), \quad j = 0, 1, \dots, m, \qquad \sigma_j^0 = 0, \quad j > m \geq 0.$$

Via (3.2) and (3.3) the sufficient condition for asymptotic mean square stability of the trivial solution of (5.21) has the form

$$|a| + |b|\frac{k(k+1)}{2} < \sqrt{1-S_0}, \quad S_0 = \frac{(m+1)^2(m+2)^2}{4}\sigma^2. \tag{5.22}$$

From (3.2) and (3.4) we obtain another sufficient condition for the asymptotic mean square stability of the trivial solution of (5.21):

$$\begin{aligned} &S_0 < \left(1 - a - b\frac{k(k+1)}{2}\right)\left(1 + a + b\frac{k(k+1)}{2} - |b|\frac{k(k+1)(k+2)}{3}\right), \\ &\left|a + b\frac{k(k+1)}{2}\right| < 1. \end{aligned} \tag{5.23}$$

Using conditions (3.11) and (3.12) we obtain a sufficient condition for the asymptotic mean square stability of the trivial solution of (5.21) in the form

$$\begin{aligned} &\frac{\frac{|b|k(k-1)}{2}[2|a| + (1-bk)|b|\frac{k(k+3)}{2}] + (1-bk)S_0}{(1+bk)[(1-bk)^2 - a^2]} < 1, \\ &|b|k < 1, \; |a| < 1 - bk. \end{aligned} \tag{5.24}$$

Let us show that using some special way of estimation of $\mathbf{E}\Delta V_{1i}$ and supposing some particular conditions on a and b we can get a sufficient stability condition, which differs from the conditions (5.22), (5.23) and (5.24) and gives an additional region of stability.

Consider the representation (1.7) for (5.21) in the form

$$\begin{aligned} &\tau = 0, \qquad F_1 = ax_i, \qquad F_2 = b\sum_{j=1}^{k}(k+1-j)x_{i-j}, \\ &F_3 = G_1 = 0, \qquad G_2 = \sigma\sum_{j=0}^{m}(m+1-j)x_{i-j}. \end{aligned} \tag{5.25}$$

Putting $V_{1i} = x_i^2$ and using the standard way of estimation $\mathbf{E}\Delta V_{1i}$ we get the sufficient stability condition in the form of (5.22). Now we will use again representation (5.25) and the functional $V_{1i} = x_i^2$, but we choose another way of estimation of $\mathbf{E}\Delta V_{1i}$.

Calculating $\mathbf{E}\Delta V_{1i}$, we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left(ax_i + b\sum_{j=1}^{k}(k+1-j)x_{i-j} + \sigma\sum_{j=0}^{m}(m+1-j)x_{i-j}\xi_{i+1}\right)^2 - \mathbf{E}x_i^2 \\
&= \big(a^2-1\big)\mathbf{E}x_i^2 + 2ab\sum_{j=1}^{k}(k+1-j)\mathbf{E}x_i x_{i-j} \\
&\quad + b^2\mathbf{E}\left(\sum_{j=1}^{k}(k+1-j)x_{i-j}\right)^2 + \sigma^2\mathbf{E}\left(\sum_{j=0}^{m}(m+1-j)x_{i-j}\right)^2 \\
&\le \big(a^2-1\big)\mathbf{E}x_i^2 + 2ab\sum_{j=1}^{k}(k+1-j)\mathbf{E}x_i x_{i-j} \\
&\quad + b^2\lambda_{k1}\sum_{j=1}^{k}(k+1-j)\mathbf{E}x_{i-j}^2 + \sigma^2\lambda_{m0}\sum_{j=0}^{m}(m+1-j)\mathbf{E}x_{i-j}^2, \qquad (5.26)
\end{aligned}$$

where

$$\lambda_{kl} = \frac{1}{2}(k+1-l)(k+2-l).$$

Via the formal procedure of Lyapunov functionals construction let us use the additional part V_{2i} of the functional $V_i = V_{1i} + V_{2i}$ in the form

$$V_{2i} = |ab|\sum_{j=1}^{k+1}\left(\sum_{l=1}^{j}x_{i-l}\right)^2.$$

Then

$$\begin{aligned}
\Delta V_{2i} &= |ab|\sum_{j=1}^{k+1}\left[\left(\sum_{l=1}^{j}x_{i+1-l}\right)^2 - \left(\sum_{l=1}^{j}x_{i-l}\right)^2\right] \\
&= |ab|\sum_{j=1}^{k+1}\left[\left(\sum_{l=0}^{j-1}x_{i-l}\right)^2 - \left(\sum_{l=1}^{j}x_{i-l}\right)^2\right] \\
&= |ab|\sum_{j=1}^{k+1}\left[\left(x_i + \sum_{l=1}^{j-1}x_{i-l}\right)^2 - \left(x_{i-j} + \sum_{l=1}^{j-1}x_{i-l}\right)^2\right] \\
&= |ab|\sum_{j=1}^{k+1}\left[x_i^2 + 2x_i\sum_{l=1}^{j-1}x_{i-l} - \left(x_{i-j}^2 + 2x_{i-j}\sum_{l=1}^{j-1}x_{i-l}\right)\right]
\end{aligned}$$

$$= |ab|\left[(k+1)x_i^2 + 2x_i \sum_{l=1}^{k}(k+1-l)x_{i-l} - \rho_i\right],$$

where

$$\rho_i = \sum_{j=1}^{k+1}\left(x_{i-j}^2 + 2x_{i-j}\sum_{l=1}^{j-1} x_{i-l}\right).$$

It is easy to see that

$$\begin{aligned}
\rho_i &= \sum_{j=1}^{k+1} x_{i-j}^2 + \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{j-1} x_{i-l} + \sum_{l=1}^{k} x_{i-l} \sum_{j=l+1}^{k+1} x_{i-j} \\
&= \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{j} x_{i-l} + \sum_{j=1}^{k+1} x_{i-j} \sum_{l=j+1}^{k+1} x_{i-l} \\
&= \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{k+1} x_{i-l} = \left(\sum_{j=1}^{k+1} x_{i-j}\right)^2 \geq 0.
\end{aligned}$$

Therefore,

$$\Delta V_{2i} \leq |ab|\left((k+1)x_i^2 + 2\sum_{l=1}^{k}(k+1-l)x_i x_{i-l}\right). \tag{5.27}$$

Let us suppose now that $ab < 0$. Then from (5.26) and (5.27) for the functional $V_i = V_{1i} + V_{2i}$ it follows that

$$\begin{aligned}
\mathbf{E}\Delta V_i &\leq \left[a^2 + |ab|(k+1) - 1\right]\mathbf{E}x_i^2 \\
&\quad + b^2\lambda_{k1}\sum_{j=1}^{k}(k+1-j)\mathbf{E}x_{i-j}^2 + \sigma^2\lambda_{m0}\sum_{j=0}^{m}(m+1-j)\mathbf{E}x_{i-j}^2.
\end{aligned}$$

Using Theorem 1.2 and S_0 from (5.22) we find that the inequality

$$a^2 + |ab|(k+1) + b^2\frac{k^2(k+1)^2}{4} + S_0 < 1, \quad ab < 0, \tag{5.28}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (5.21).

It is easy to see that for $ab < 0$ condition (5.28) is weaker than condition (5.22) if $k > 1$ and coincides with condition (5.22) if $k = 1$.

In Fig. 5.4 the stability regions, which are obtained by conditions (5.22), (5.23), (5.24) and (5.28), are shown in the (a, b)-plane for $k = 2$, $S_0 = 0.1$ (the bounds with numbers 1, 2, 3, 4, respectively).

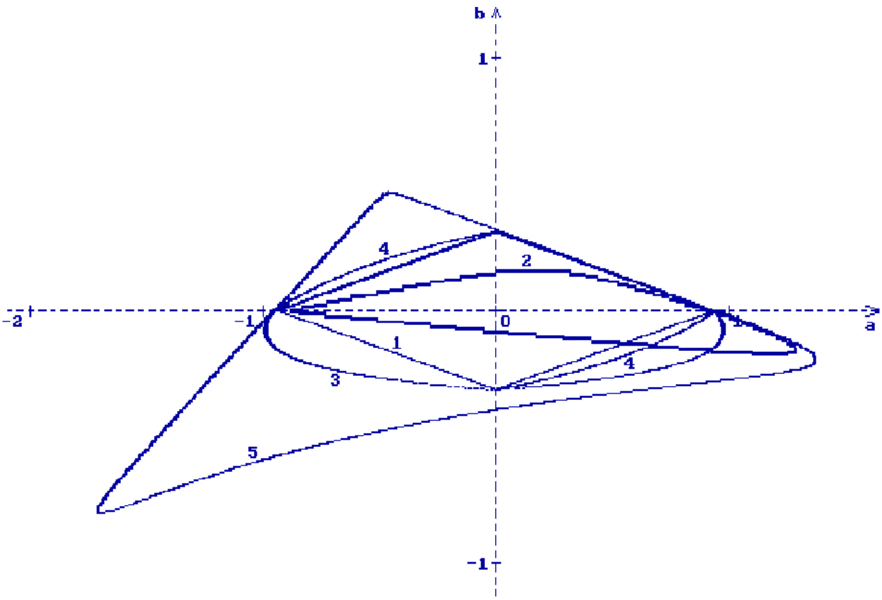


Fig. 5.4 Stability regions for (5.21) given for $k = 2$, $S_0 = 0.1$, by conditions: (*1*) (5.22), (*2*) (5.23), (*3*) (5.24), (*4*) (5.28), (*5*) (5.29)

Note that if $k = 2$ then (5.21) is a particular case of (5.8) for $a_0 = a$, $a_1 = 2b$, $a_2 = b$, $m = 0$. In this case using the stability condition (5.9) for (5.8), we obtain the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.21) in the form

$$\sigma^2 < 1 - a^2 - 5b^2 - 4ab^2 - \frac{2b(a+b)(a+2b^2)(a+2)}{1 - b(2+a+b)}. \tag{5.29}$$

The corresponding stability region is shown in Fig. 5.4 (the bound with number 5).

5.3 Volterra Equations

In this section two examples of Volterra equations are investigated in detail.

Example 5.2 Consider the equation

$$x_{i+1} = ax_i + \sum_{j=1}^{i+h} b^j x_{i-j} + \sigma x_{i-m}\xi_{i+1}, \quad m \geq 0,\ i = 0, 1, \ldots. \tag{5.30}$$

Using (3.3), (3.13) and (3.25) (or in accordance with Remark 3.4 condition (3.39) for $k = 0$, $k = 1$ and $k = 2$) we obtain the following sufficient conditions for asymptotic mean square stability of the trivial solution of (5.30):

$$|a| + \frac{|b|}{1-|b|} < \sqrt{1-\sigma^2}, \quad |b| < 1, \tag{5.31}$$

$$\frac{b^2}{1-|b|}\left(\frac{|b|(2-|b|)}{1-|b|} + \frac{2|a|}{1-b}\right) + \sigma^2 < 1 - b^2 - a^2\frac{1+b}{1-b},$$
$$|a| + b < 1, \ |b| < 1, \tag{5.32}$$

$$\frac{|b|^6}{(1-|b|)^2} + \frac{2|b|^3}{1-|b|}\left(b^2 + \frac{|a+b^3| + (1-b)|b(1+ab)|}{|1-b-b^2(a+b^2)|}\right) + \sigma^2$$
$$< 1 - a^2 - b^2 - b^4 - 2ab^3 - 2b\frac{(1+ab)(a+b^2)(a+b^3)}{1-b-b^2(a+b^2)}, \quad |b| < 1. \tag{5.33}$$

Using the program "Mathematica" for the solution of the matrix equation (3.29), the sufficient condition (3.39) for the asymptotic mean square stability of the trivial solution of (5.30) was obtained also for $k = 3$ and $k = 4$. In particular, for $k = 3$ this condition takes the form

$$\frac{b^4}{1-|b|} < \sqrt{\beta_3^2 + d_{44}^{-1} - \sigma^2} - \beta_3, \quad |b| < 1,$$
$$\beta_3 = |b^3| + \left|a + \frac{d_{34}}{d_{44}}\right| + \left|b + \frac{d_{24}}{d_{44}}\right| + \left|b^2 + \frac{d_{14}}{d_{44}}\right|, \tag{5.34}$$

where

$$\frac{d_{14}}{d_{44}} = b^3\left[b^3 + b^5 - b^8 + a\left(1 - b^3 + b^4\right)\right]G^{-1},$$

$$\frac{d_{24}}{d_{44}} = b^2\left[a^2 b + b^2 + b^5 - b^6 - b^8 + a\left(1 + b^4 + b^6\right)\right]G^{-1},$$

$$\frac{d_{34}}{d_{44}} = b\left[b^2 + a^3 b^2 + b^4 - b^7 + a^2\left(b + b^4\right) + a\left(1 + 2b^3 + b^5 - b^6 - b^8\right)\right]G^{-1},$$

$$d_{44} = G\big[1 - b - b^2 - a^4 b^3 - 2b^4 + 2b^7 - 2b^8 + 2b^9 - b^{10} - b^{12} + b^{13}$$
$$- b^{14} + b^{17} - a^3\left(b^2 + b^5\right) - a^2\left(1 + b + 5b^4 - b^5 + b^6 - 2b^7 - b^9\right)$$
$$- ab^2\left(1 + 4b - b^2 + 5b^3 - b^4 + b^5 - 4b^6 + 4b^7 - b^{10} + b^{11}\right)\big]^{-1},$$

$$G = 1 - b - ab^2 - \left(1 + a^2\right)b^3 - b^4 - ab^5 - b^6 + b^7 + b^9.$$

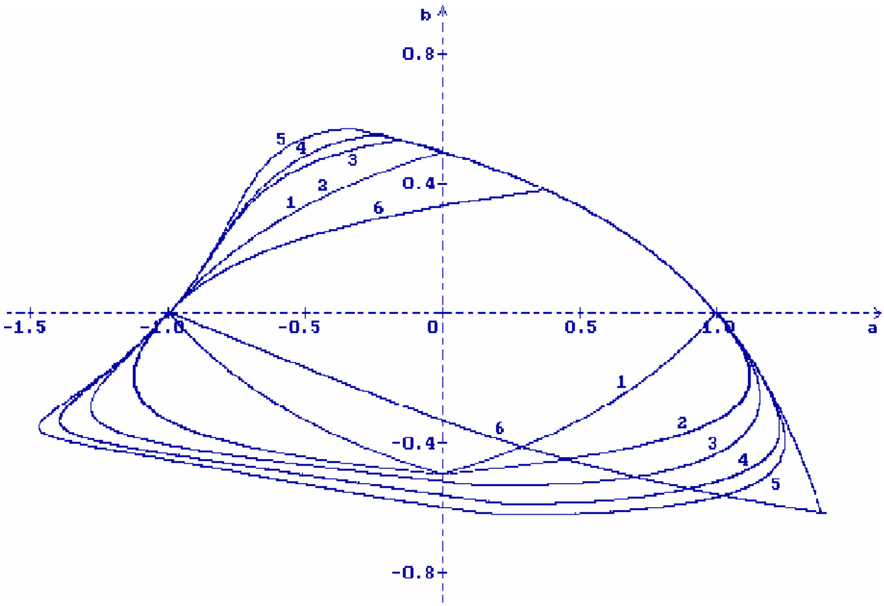


Fig. 5.5 Stability regions for (5.30) given for $\sigma^2 = 0$ by (*1*) $k = 0$, condition (5.31), (*2*) $k = 1$, condition (5.32), (*3*) $k = 2$, condition (5.33), (*4*) $k = 3$, condition (5.34), (*5*) $k = 4$, (*6*) condition (5.35)

Condition (3.4) for (5.30) takes the form

$$-\frac{1-3|b|}{(1-b)(1-|b|)} < a < \frac{1-2b}{1-b}, \quad |b| < 1,$$
$$\sigma^2 < \left(\frac{1-2b}{1-b} - a\right)\left(a + \frac{1-3|b|}{(1-b)(1-|b|)}\right). \tag{5.35}$$

In Fig. 5.5 the regions of asymptotic mean square stability of the trivial solution of (5.30) are shown, obtained for $\sigma^2 = 0$ and $k = 0$ (condition (5.31), curve number 1), $k = 1$ (condition (5.32), curve number 2), $k = 2$ (condition (5.33), curve number 3), $k = 3$ (condition (5.34), curve number 4), $k = 4$ (curve number 5) and also the region obtained by condition (5.35) (curve number 6). In Fig. 5.6 and Fig. 5.7 the similar regions of stability are shown for $\sigma^2 = 0.3$ and $\sigma^2 = 0.7$.

In Fig. 5.5 one can see (and naturally it can be shown analytically) that in the case $\sigma^2 = 0$ for $b \geq 0$ the stability condition (5.31) coincides with the condition (5.32) and for $a \geq 0$, $b \geq 0$ the stability conditions for $k = 0, 1, 2, 3, 4$ give the same region of asymptotic mean square stability, which is defined by the inequality

$$a + \frac{b}{1-b} < 1, \quad b < 1.$$

According to Remark 3.3 in Figs. 5.5–5.7 one can see also that the region of asymptotic mean square stability Q_k of the trivial solution of (5.30), obtained by condi-

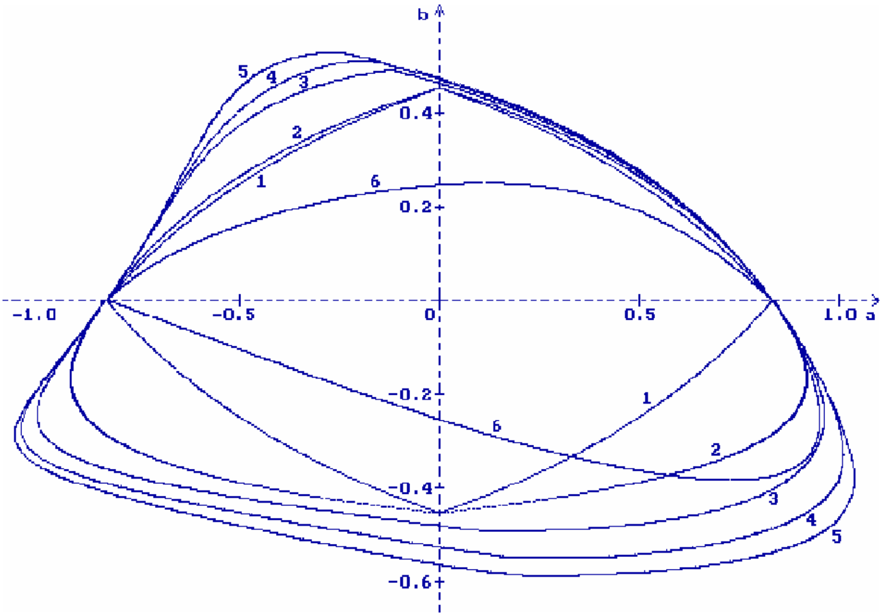


Fig. 5.6 Stability regions for (5.30) given for $\sigma^2 = 0.3$ by (*1*) $k = 0$, condition (5.31), (*2*) $k = 1$, condition (5.32), (*3*) $k = 2$, condition (5.33), (*4*) $k = 3$, condition (5.34), (*5*) $k = 4$, (*6*) condition (5.35)

tion (3.39), expands if k increases, i.e. $Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset Q_4$. Thus, to get a greater region of asymptotic mean square stability one can use the condition (3.39) for $k = 5$, $k = 6$ etc. But on the other hand it is clear that each region Q_k can be obtained by the condition $|b| < 1$ only.

To obtain a condition of another type for the asymptotic mean square stability of the trivial solution of (5.30), transform the sum in (5.30) for $i \geq 1$ in the following way:

$$\begin{aligned}\sum_{j=1}^{i+h} b^j x_{i-j} &= b\left(x_{i-1} + \sum_{j=1}^{i-1+h} b^j x_{i-1-j}\right)\\ &= b\big[(1-a)x_{i-1} + x_i - \sigma x_{i-1-m}\xi_i\big].\end{aligned} \tag{5.36}$$

Substituting (5.36) into (5.30) we transform (5.30) into the form

$$\begin{aligned}x_1 &= a x_0 + \sum_{j=1}^{h} b^j x_{-j} + \sigma x_{-m}\xi_1,\\ x_{i+1} &= a_0 x_i + a_1 x_{i-1} + \sigma_1 x_{i-m}\xi_{i+1} + \sigma_2 x_{i-1-m}\xi_i, \quad i = 1, 2, \ldots,\end{aligned} \tag{5.37}$$

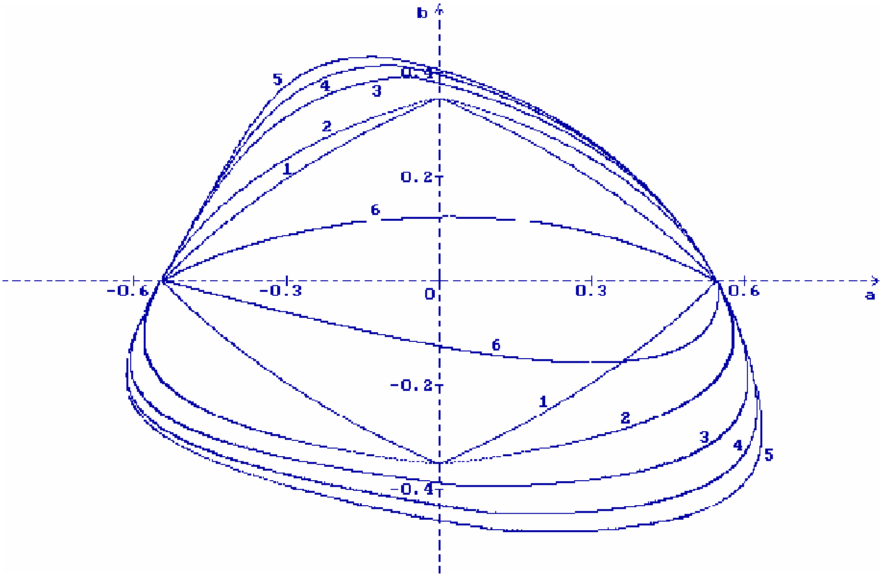


Fig. 5.7 Stability regions for (5.30) given for $\sigma^2 = 0.7$ by (*1*) $k = 0$, condition (5.31), (*2*) $k = 1$, condition (5.32), (*3*) $k = 2$, condition (5.33), (*4*) $k = 3$, condition (5.34), (*5*) $k = 4$, (*6*) condition (5.35)

with

$$a_0 = a + b, \qquad a_1 = b(1 - a), \qquad \sigma_1 = \sigma, \qquad \sigma_2 = -b\sigma. \tag{5.38}$$

We will construct the Lyapunov functional for (5.37) in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$. Then for $i \geq 1$ we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[\big(a_0 x_i + a_1 x_{i-1} + \sigma_1 x_{i-m}\xi_{i+1} + \sigma_2 x_{i-1-m}\xi_i\big)^2 - x_i^2\big] \\
&= \mathbf{E}\big[\big(a_0^2 - 1\big)x_i^2 + a_1^2 x_{i-1}^2 + \sigma_1^2 x_{i-m}^2 + \sigma_2^2 x_{i-1-m}^2 \\
&\quad + 2a_0 a_1 x_i x_{i-1}\big] + 2a_0\sigma_2 \mathbf{E} x_i x_{i-1-m}\xi_i \\
&\leq \mathbf{E}\big[\big(a_0^2 + |a_0 a_1| - 1\big)x_i^2 + \big(a_1^2 + |a_0 a_1|\big)x_{i-1}^2 \\
&\quad + \sigma_1^2 x_{i-m}^2 + \sigma_2^2 x_{i-1-m}^2\big] + 2a_0\sigma_2 \mathbf{E} x_i x_{i-1-m}\xi_i.
\end{aligned}$$

Note that via (5.37)

$$\begin{aligned}
\mathbf{E} x_i x_{i-1-m}\xi_i &= \mathbf{E} x_{i-1-m}\big(a_0 x_{i-1} + a_1 x_{i-2} + \sigma_1 x_{i-1-m}\xi_i + \sigma_2 x_{i-2-m}\xi_{i-1}\big)\xi_i \\
&= \sigma_1 \mathbf{E} x_{i-1-m}^2.
\end{aligned} \tag{5.39}$$

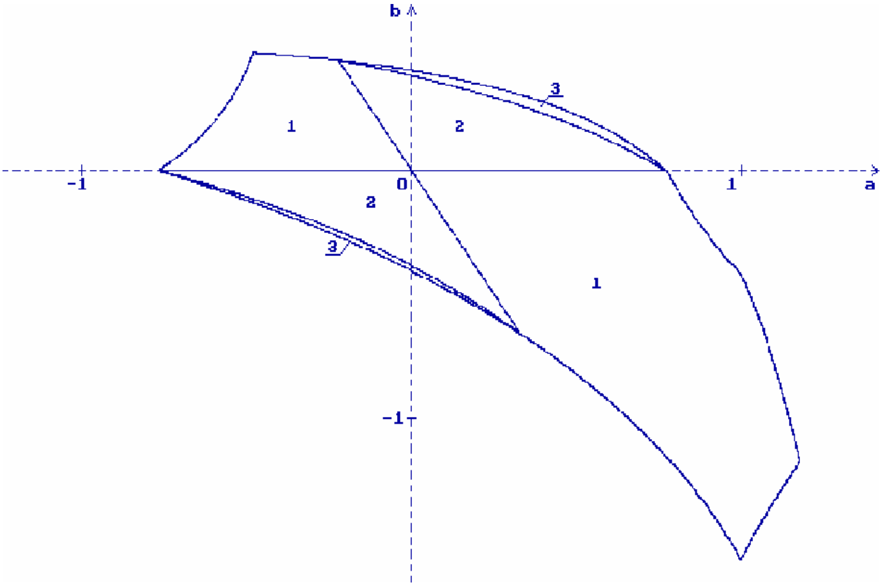


Fig. 5.8 Stability regions for (5.30) given for $\sigma^2 = 0.4$ by condition (5.42): (*1*) R_0, (*2*) R_1, (*3*) R_2

Putting $\rho = -(2a + b)b$, from (5.38) and (5.39) we obtain

$$\mathbf{E}\Delta V_{1i} \le \big(a_0^2 + |a_0a_1| - 1\big)\mathbf{E}x_i^2 + \big(a_1^2 + |a_0a_1|\big)\mathbf{E}x_{i-1}^2 \\ + \sigma^2\mathbf{E}x_{i-m}^2 + \rho\sigma^2\mathbf{E}x_{i-1-m}^2. \tag{5.40}$$

From (5.40) and Theorem 1.2 it follows that in the region $R_0 = \{\rho \ge 0\}$ the inequality

$$\big(|a_0| + |a_1|\big)^2 + (1 + \rho)\sigma^2 < 1 \tag{5.41}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (5.30). From Theorem 1.4 it follows also that by condition (5.41) in the region $R_1 = \{\rho < 0, (|a_0| + |a_1|)^2 + \sigma^2 \le 1\}$ the trivial solution of (5.30) is asymptotically mean square stable. In the region $R_2 = \{\rho < 0, (|a_0| + |a_1|)^2 + \sigma^2 > 1\}$ we can conclude only that by condition (5.41) each mean square bounded solution of (5.30) is asymptotically mean square trivial.

Note that via (5.38) and the representation for ρ, the condition (5.41) can be written in the form

$$\big(|a + b| + \big|b(1 - a)\big|\big)^2 + \sigma^2\big(1 - (2a + b)b\big) < 1. \tag{5.42}$$

In Fig. 5.8 the region given by condition (5.42) for $\sigma^2 = 0.4$ and also the following different parts of this region: (1) R_0, (2) R_1, (3) R_2, are shown.

Rewrite (5.37) in the form

$$x(i + 1) = Ax(i) + B(i), \tag{5.43}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ a_1 & a_0 \end{pmatrix}, \qquad x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \qquad B(i) = \begin{pmatrix} 0 \\ b(i) \end{pmatrix},$$
$$b(i) = \sigma_1 x_{i-m}\xi_{i+1} + \sigma_2 x_{i-1-m}\xi_i. \tag{5.44}$$

Consider now the functional $V_{1i} = x'(i)Dx(i)$, where the matrix D is the solution of (3.10) and has the elements

$$d_{11} = a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1-a_1} d_{22},$$
$$d_{22} = \frac{1-a_1}{(1+a_1)[(1-a_1)^2 - a_0^2]}. \tag{5.45}$$

The matrix D with elements (5.45) is positive semidefinite with $d_{22} > 0$ if and only if

$$|a_1| < 1, \qquad |a_0| < 1 - a_1. \tag{5.46}$$

Calculating $\mathbf{E}\Delta V_{i1}$ via (5.43) and (3.10) we have

$$\begin{aligned} \mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big] \\ &= \mathbf{E}\big[\big(Ax(i) + B(i)\big)' D\big(Ax(i) + B(i)\big) - x'(i)Dx(i)\big] \\ &= -\mathbf{E}x_i^2 + \mathbf{E}B'(i)DB(i) + 2\mathbf{E}B'(i)DAx(i) \\ &= -\mathbf{E}x_i^2 + d_{22}\mathbf{E}b_i^2 + 2\mathbf{E}b(i)\big[d_{22}a_1 x_{i-1} + (d_{12} + d_{22}a_0)x_i\big]. \end{aligned}$$

Using the representations (5.37), (5.44) and (5.45) for x_i, $b(i)$ and d_{12}, we obtain

$$\mathbf{E}b^2(i) = \sigma_1^2 \mathbf{E}x_{i-m}^2 + \sigma_2^2 \mathbf{E}x_{i-1-m}^2$$

and

$$\begin{aligned} &\mathbf{E}b(i)\big[d_{22}a_1 x_{i-1} + (d_{12} + d_{22}a_0)x_i\big] \\ &\quad = (d_{12} + d_{22}a_0)\sigma_1\sigma_2 \mathbf{E}x_{i-1-m}^2 \\ &\quad = \frac{a_0}{1-a_1}\sigma_1\sigma_2 d_{22}\mathbf{E}x_{i-1-m}^2. \end{aligned}$$

Thus,

$$\mathbf{E}\Delta V_{1i} = -\mathbf{E}x_i^2 + \sigma_1^2 d_{22}\mathbf{E}x_{i-m}^2 + \gamma d_{22}\mathbf{E}x_{i-1-m}^2,$$

where

$$\gamma = \frac{2a_0}{1-a_1}\sigma_1\sigma_2 + \sigma_2^2. \tag{5.47}$$

Put now

$$V_{2i}=\sigma_1^2 d_{22}\sum_{j=1}^{m}x_{i-j}^2.$$

Then $\Delta V_{2i}=\sigma_1^2 d_{22}(x_i^2-x_{i-m}^2)$ and for the functional $V_i=V_{1i}+V_{2i}$ we have

$$\mathbf{E}\Delta V_i=-\big(1-\sigma_1^2 d_{22}\big)\mathbf{E}x_i^2+\gamma d_{22}\mathbf{E}x_{i-1-m}^2. \tag{5.48}$$

Note that by condition (5.46) $\sigma_1^2+\gamma>0$. In fact,

$$\begin{aligned}\sigma_1^2+\gamma&=\sigma_1^2+\frac{2a_0}{1-a_1}\sigma_1\sigma_2+\sigma_2^2\\&\geq\sigma_1^2-\frac{2|a_0|}{1-a_1}|\sigma_1\sigma_2|+\sigma_2^2\\&>\sigma_1^2-2|\sigma_1\sigma_2|+\sigma_2^2=\big(|\sigma_1|-|\sigma_2|\big)^2\geq 0.\end{aligned}$$

From Corollary 1.1 it follows that in the region $G_0=\{\gamma\geq 0\}$ the inequality

$$\big(\sigma_1^2+\gamma\big)d_{22}<1 \tag{5.49}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.30). From Theorem 1.4 it follows also that by condition (5.49) in the region $G_1=\{\gamma<0,\sigma_1^2 d_{22}\leq 1\}$ the trivial solution of (5.30) is asymptotically mean square stable. In the region $G_2=\{\gamma<0,\sigma_1^2 d_{22}>1\}$ we can conclude only that by condition (5.49) each mean square bounded solution of (5.30) is asymptotically mean square trivial. In Fig. 5.9 the region is shown given by the condition (5.49) for $\sigma^2=0.3$, and also the different parts of this region (1) G_0, (2) G_1, (3) G_2 are shown.

Note that in the region G_2 the trivial solution of (5.30) can be asymptotically mean square stable too. In fact, in Fig. 5.10 the regions of asymptotic mean square stability of the trivial solution of (5.30) for $\sigma^2=0.3$ from Fig. 5.6 are shown together with the region G_2 from Fig. 5.9: $k=0$ (condition (5.31), curve number 1), $k=1$ (condition (5.32), curve number 2), $k=2$ (condition (5.33), curve number 3), $k=3$ (condition (5.34), curve number 4), $k=4$ (curve number 5), the region obtained by condition (5.35) (curve number 6), the region obtained by condition (5.41) (curve number 7), the region obtained by condition (5.49) (curve number 8), the region G_2 (between curves 8 and 9). It is easy to see that most of the region obtained by condition (5.41) and some part of the region G_2 belong to the regions where the trivial solution of (5.30) is asymptotically mean square stable.

Example 5.3 Consider the equation

$$x_{i+1}=ax_i+b\sum_{j=-h}^{i-1}x_j+\sigma x_{i-m}\xi_{i+1}. \tag{5.50}$$

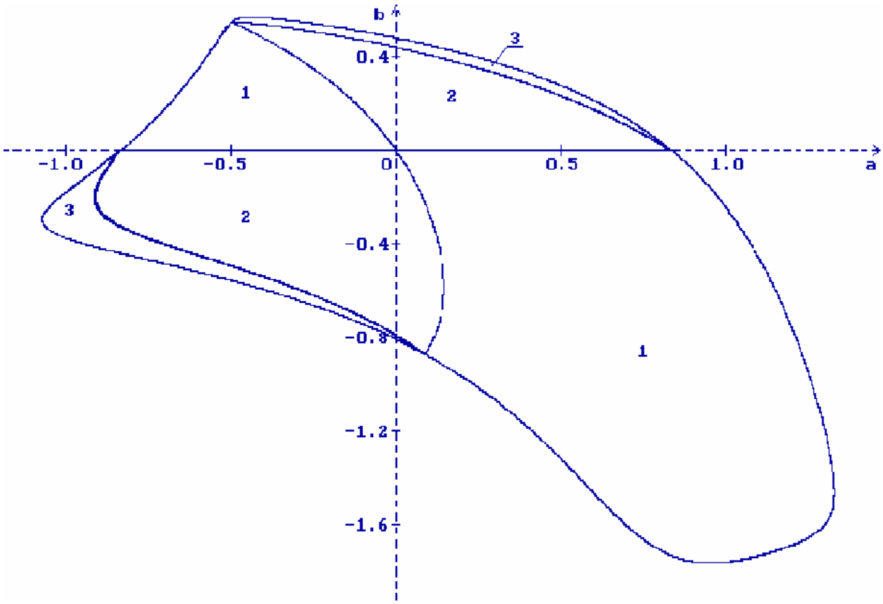


Fig. 5.9 Stability regions for (5.30) given for $\sigma^2 = 0.3$ by condition (5.49): (*1*) G_0, (*2*) G_1, (*3*) G_2

From (4.33) it follows that a sufficient condition for asymptotic mean square stability of the trivial solution of (5.50) has the form

$$0 \geq b \geq a > \frac{b}{2} - 1 + \sigma^2. \tag{5.51}$$

To get stability conditions of another type put $\beta = 1$ if $b \neq 0$ and $\beta = 0$ if $b = 0$ and transform (5.50) for $i \geq 1$ in the following way:

$$\begin{aligned} x_{i+1} &= ax_i + bx_{i-1} + b \sum_{j=-h}^{i-2} x_j + \sigma x_{i-m}\xi_{i+1} \\ &= ax_i + bx_{i-1} + \beta(x_i - ax_{i-1} - \sigma x_{i-1-m}\xi_i) + \sigma x_{i-m}\xi_{i+1}. \end{aligned}$$

Similar to (5.37) we obtain (5.50) in the form

$$\begin{aligned} x_1 &= ax_0 + b\sum_{j=1}^{h} x_{-j} + \sigma x_{-m}\xi_1, \\ x_{i+1} &= a_0 x_i + a_1 x_{i-1} + \sigma_1 x_{i-m}\xi_{i+1} + \sigma_2 x_{i-1-m}\xi_i, \quad i = 1, 2, \ldots, \end{aligned} \tag{5.52}$$

with

$$a_0 = a + \beta, \qquad a_1 = b - \beta a, \qquad \sigma_1 = \sigma, \qquad \sigma_2 = -\beta\sigma. \tag{5.53}$$

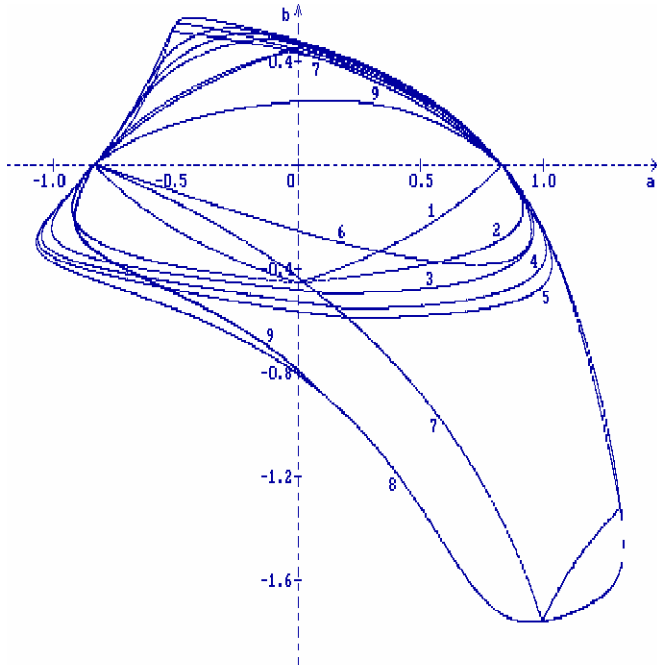


Fig. 5.10 Stability regions for (5.30), given for $\sigma^2 = 0.3$ by: (*1*) $k = 0$, condition (5.31), (*2*) $k = 1$, condition (5.32), (*3*) $k = 2$, condition (5.33), (*4*) $k = 3$, condition (5.34), (*5*) $k = 4$, (*6*) condition (5.35), (*7*) condition (5.41), (*8*) condition (5.49), (*9*) G_2 between curves *8* and *9*

Therefore, for (5.52) there exists a functional V_i satisfying condition (5.40) with $\rho = -(2a + \beta)\beta$. If $\rho \geq 0$, then the inequality (5.41) is a sufficient condition for asymptotic mean square stability of the trivial solution of (5.50). In particular, if $b = 0$ then $\beta = 0$ and the condition (5.41) takes the form $a^2 + \sigma^2 < 1$, which is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.50) with $b = 0$. If $b \neq 0$ then $\beta = 1$, and the condition $\rho \geq 0$ is equivalent to $a \leq -0.5$.

From Theorem 1.4 it follows also that by condition (5.41) in the region $Q_1 = \{a > -0.5, (|a_0| + |a_1|)^2 + \sigma^2 \leq 1\}$ the trivial solution of (5.50) is asymptotically mean square stable. In the region $Q_2 = \{a > -0.5, (|a_0| + |a_1|)^2 + \sigma^2 > 1\}$ we can conclude only that by condition (5.41) each mean square bounded solution of (5.50) is asymptotically mean square trivial.

Note that for $b \neq 0$ from (5.53) and the representation for ρ it follows that the condition (5.41) can be written in the form

$$\big(|a + 1| + |b - a|\big)^2 - 2a\sigma^2 < 1. \tag{5.54}$$

In Fig. 5.11 the region given by condition (5.54) for $\sigma^2 = 0.2$ and also the different parts of this region $Q_0 = \{a \leq -0.5\}$ (region ABCDFA), Q_1 (region DEFD), and Q_2 (region DEFOD) are shown. For comparison the region given by condition

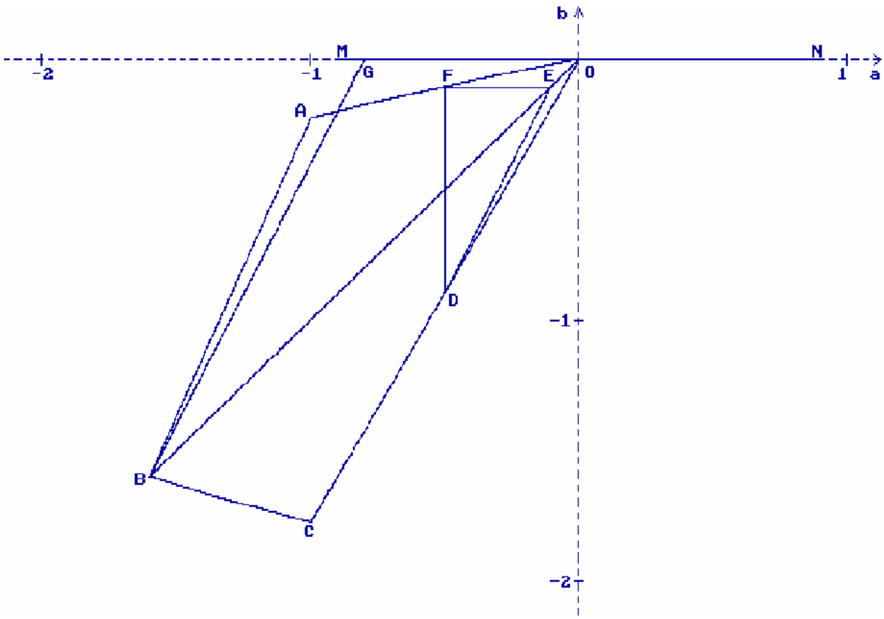


Fig. 5.11 Stability regions for (5.50) given by condition (5.54) for $\sigma^2 = 0.2$

(5.51) is shown too (region GBOG). Using Theorem 1.4 one can assert that in the region ABCDEFA the trivial solution of (5.50) is asymptotically mean square stable, but in the region DEFOD we can conclude only that each mean square bounded solution of (5.50) is asymptotically mean square trivial. On the other hand one can see that the part EFOE of region DEFOD belongs to region GBOG, and therefore in region EFOE as well as in the whole region GBOG the trivial solution of (5.50) is asymptotically mean square stable. If $b = 0$ then the line segment MN is the region of the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.50).

To get one more condition, note that (5.52) coincides with (5.37) for $i \geq 1$. Therefore, for (5.52) there exists a functional V_i that satisfies condition (5.48) with γ defined by (5.47). It means that the inequality (5.49) gives the stability condition for (5.50) also.

Consider condition (5.49) for (5.50) in more detail. Note that if $b = 0$ then $\beta = 0$. Via (5.45) and (5.53) in this case condition (5.49) takes the form

$$|a| < \sqrt{1 - \sigma^2} \tag{5.55}$$

and is the necessary and sufficient condition for mean square stability of the trivial solution of (5.48).

Let $b \neq 0$. In this case $\beta = 1$, and via (5.46) and (5.53) we have $d_{22} > 0$ if and only if

$$|b - a| < 1, \qquad |a + 1| < 1 + a - b. \tag{5.56}$$

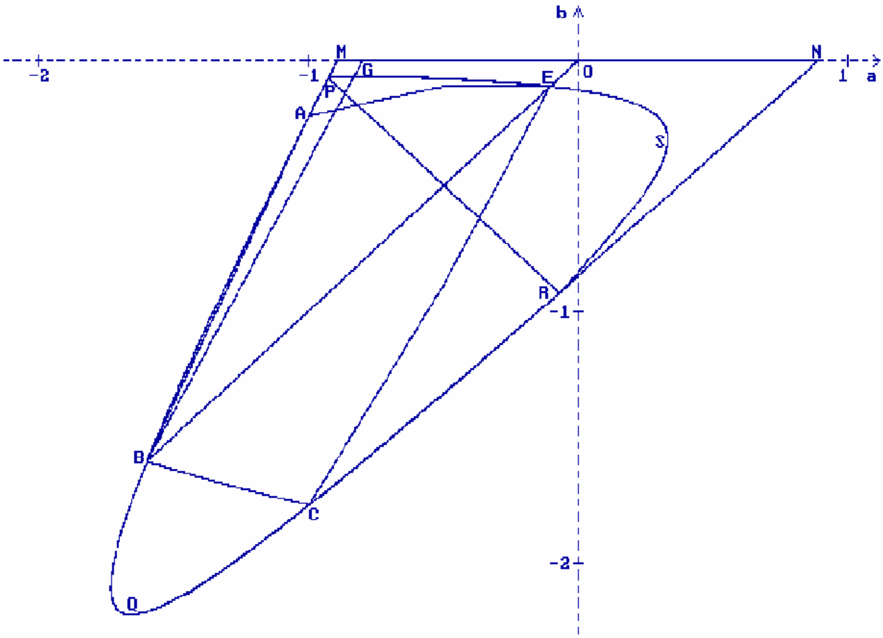


Fig. 5.12 Stability regions for (5.50) given by condition (5.58) for $\sigma^2 = 0.2$

Inequalities (5.56) are equivalent to the condition

$$a - 1 < b < \begin{cases} 2(a+1), & a \in (-3, -1), \\ 0, & a \in [-1, 1). \end{cases} \tag{5.57}$$

In the deterministic case ($\sigma = 0$) condition (5.57) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.50).

Via (5.53) the condition (5.49) can be written in the form

$$\sigma^2 d_{22}(1+\delta) < 1 \tag{5.58}$$

with

$$\delta = -\frac{1+a+b}{1+a-b}. \tag{5.59}$$

It is easy to see that by condition (5.57) $1 + \delta > 0$. Via Corollary 1.1 one can show that if $\delta \geq 0$ then the inequality (5.58) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (5.50).

From Theorem 1.4 it follows also that by the condition (5.58) in the region $\{\delta < 0, \sigma^2 d_{22} \leq 1\}$ the trivial solution of (5.50) is asymptotically mean square stable. In the region $\{\delta < 0, \sigma^2 d_{22} > 1\}$ we can conclude only that by condition (5.58) each mean square bounded solution of (5.50) is asymptotically mean square trivial.

In Fig. 5.12 the region given by condition (5.58) for $\sigma^2 = 0.2$, and also the different parts of this region PQRP $\{\delta \geq 0\}$, PRSP $\{\delta < 0,\ \sigma^2 d_{22} \leq 1\}$, PMNRSP $\{\delta < 0,\ \sigma^2 d_{22} > 1\}$ are shown.

In reality in the region $\{\delta < 0,\ \sigma^2 d_{22} > 1\}$ the trivial solution of (5.50) can be asymptotically mean square stable too. In fact, in Fig. 5.12 one can see that part of the region $\{\delta < 0,\ \sigma^2 d_{22} > 1\}$ belongs to the region GBOG of asymptotic mean square stability of the trivial solution of (5.50) given by condition (5.51).

In Fig. 5.12 it is shown also that the region PQRSP, where the trivial solution of (5.50) is asymptotically mean square stable, includes the region of asymptotic mean square stability ABCEA that was obtained by virtue of condition (5.54).

5.4 Difference Equation with Markovian Switching

The stability of stochastic differential equations with Markovian switching has received a great deal of attention (see [182, 230, 237] and references therein). Consider the following stochastic difference equation with Markovian switching

$$x_{i+1} = \eta_i x_i + b x_{i-h} + \sigma x_{i-l} \xi_{i+1}, \quad i \in Z. \tag{5.60}$$

Here ξ_i is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, η_i is $\mathfrak{F}_i$-adapted and independent on the ξ_i Markov chain [87–89] with denumerable set of states $\alpha_1, \alpha_2, \dots$ and probabilities of transition

$$P_{mk} = \mathbf{P}\{\eta_{i+1} = \alpha_k / \eta_i = \alpha_m\}. \tag{5.61}$$

First consider the auxiliary equation without delays

$$y_{i+1} = \eta_i y_i. \tag{5.62}$$

Consider also the auxiliary Markov chain ζ_i with denumerable set of states $\beta_1, \beta_2, \dots$ and the same as for the η_i probabilities of transition

$$P_{mk} = \mathbf{P}\{\zeta_{i+1} = \beta_k / \zeta_i = \beta_m\}. \tag{5.63}$$

It is supposed here that ζ_i is $\mathfrak{F}_i$-adapted, independent on η_i, $\mathbf{P}\{\zeta_i = \beta_m\} = \mathbf{P}\{\eta_i = \alpha_m\}$ and $\beta = \inf_{k \geq 1} \beta_k > 0$.

Consider the function $v_i = \zeta_i y_i^2$ and calculate $\mathbf{E}\Delta v_i$. Via (5.62) and independence η_i and ζ_i on y_i we have

$$\mathbf{E}\Delta v_i = \mathbf{E}\big(\zeta_{i+1} y_{i+1}^2 - \zeta_i y_i^2\big) = \mathbf{E}\big(\zeta_{i+1}\eta_i^2 - \zeta_i\big) y_i^2 = \gamma_i \mathbf{E} y_i^2,$$

where

$$\gamma_i = \mathbf{E}\big(\zeta_{i+1}\eta_i^2 - \zeta_i\big). \tag{5.64}$$

If $\sup_{i \in Z} \gamma_i \leq -c < 0$ then $\mathbf{E}\Delta v_i \leq -c\mathbf{E} y_i^2$ and the trivial solution of (5.50) is asymptotically mean square stable.

Remark 5.1 Let the Markov chain η_i satisfy the condition $0 < \alpha_0 \le |\eta_i| \le \alpha_1 < 1$. Then choosing $\zeta_i = |\eta_i|$ we obtain $\gamma_i = \mathbf{E}(|\eta_{i+1}|\eta_i^2 - |\eta_i|) = \mathbf{E}(|\eta_{i+1}||\eta_i| - 1)|\eta_i| \le (\alpha_1^2 - 1)\alpha_0 < 0$. It means that the trivial solution of (5.62) is asymptotically mean square stable.

Remark 5.2 Let the Markov chain η_i satisfy the condition $|\eta_i| \ge 1$. Then putting $\zeta_i = |\eta_i|$ we obtain $\gamma_i = \mathbf{E}(|\eta_{i+1}||\eta_i| - 1)|\eta_i| \ge 0$. It means that the trivial solution of (5.62) cannot be asymptotically mean square stable.

Lemma 5.1 *Let $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ be, respectively, the states of Markov chains η_i and ζ_i with the probabilities of transition* (5.61) *and* (5.63). *Then* $\sup_{i \in Z} \gamma_i \le \gamma$, *where γ_i is defined by* (5.64) *and*

$$\gamma = \sup_{m \ge 1} \left\{ \alpha_m^2 \sum_{k=1}^{\infty} P_{mk} \beta_k - \beta_m \right\}. \tag{5.65}$$

Proof Via (5.64) we have

$$\begin{aligned} \gamma_i &= \mathbf{E}\big(\eta_i^2 \mathbf{E}\{\zeta_{i+1}/\mathfrak{F}_i\} - \zeta_i\big) \\ &= \mathbf{E}\left(\eta_i^2 \sum_{k=1}^{\infty} \beta_k \mathbf{P}\{\zeta_{i+1} = \beta_k/\mathfrak{F}_i\} - \zeta_i\right). \end{aligned} \tag{5.66}$$

Since ζ_i is a Markov chain, $\mathbf{P}\{\zeta_{i+1} = \beta_k/\mathfrak{F}_i\} = \mathbf{P}\{\zeta_{i+1} = \beta_k/\zeta_i\}$. From this via (5.51) and (5.65) and (5.66) we obtain

$$\begin{aligned} \gamma_i &= \mathbf{E}\left(\eta_i^2 \sum_{k=1}^{\infty} \beta_k \mathbf{P}\{\zeta_{i+1} = \beta_k/\zeta_i\} - \zeta_i\right) \\ &= \sum_{m=1}^{\infty} \left(\alpha_m^2 \sum_{k=1}^{\infty} \beta_k \mathbf{P}\{\zeta_{i+1} = \beta_k/\zeta_i = \beta_m\} - \beta_m\right) \mathbf{P}\{\zeta_i = \beta_m\} \\ &= \sum_{m=1}^{\infty} \left(\alpha_m^2 \sum_{k=1}^{\infty} P_{mk} \beta_k - \beta_m\right) \mathbf{P}\{\zeta_i = \beta_m\} \le \gamma. \end{aligned} \tag{5.67}$$

The proof is completed. □

Corollary 5.1 *If for given $\alpha_1, \alpha_2, \dots$ and P_{mk} there exists a sequence $\beta_1, \beta_2, \dots$, such that $\gamma < 0$, then the trivial solution of* (5.62) *is asymptotically mean square stable.*

Suppose that the states of Markov chain η_i satisfy the condition: for some $k \ge 1$

$$|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_k| \ge 1 > |\alpha_{k+1}| \ge \cdots \ge \alpha > 0. \tag{5.68}$$

Put $A_k = \{\alpha_1, \dots, \alpha_k\}$,

$$Q_{mk} = \mathbf{P}\{\eta_{i+1} \in A_k / \eta_i = \alpha_m\} = \sum_{l=1}^{k} P_{ml}.$$

If $m \le k$ *then* Q_{mk} *is the probability to remain in the set* A_k; *if* $m > k$ *then* Q_{mk} *is the probability to come into the set* A_k. *The set* A_k *is a "bad" one for stability.*

Lemma 5.2 *If* $\gamma < 0$, *i.e. the trivial solution of* (5.62) *is asymptotically mean square stable, then the probability* Q_{mk} *is small enough. In particular, if* $\gamma < 0$ *for some sequence* $\beta_1, \beta_2, \dots$, *such that*

$$\hat{\beta} = \inf_{m \le k} \beta_m > \beta = \inf_{m \ge 1} \beta_m > 0, \tag{5.69}$$

then

$$Q_{mk} < \frac{\beta_m \alpha_m^{-2} - \beta}{\hat{\beta} - \beta}, \quad m = 1, 2, \dots. \tag{5.70}$$

Proof If $\gamma < 0$ then via (5.65)

$$\begin{aligned}\beta_m &> \alpha_m^2 \left(\sum_{l=1}^{k} P_{ml} \beta_l + \sum_{l=k+1}^{\infty} P_{ml} \beta_l \right) \\ &\ge \alpha_m^2 \big(\hat{\beta} Q_{mk} + \beta (1 - Q_{mk}) \big) = \alpha_m^2 \big((\hat{\beta} - \beta) Q_{mk} + \beta \big).\end{aligned}$$

From this (5.70) follows. The proof is completed. □

Remark 5.3 Choosing $\beta_m = \alpha_m^2$ from (5.68)–(5.70) we obtain

$$Q_{mk} < \frac{1 - \alpha^2}{\alpha_k^2 - \alpha^2}. \tag{5.71}$$

In this case the estimation of Q_{mk} does not depend on m.

Remark 5.4 Suppose that $\alpha < |\alpha_m^{-1}| \le |\alpha_k|$ for some $m \ge 1$. Choosing $\beta_m = |\alpha_m|$ from (5.68)–(5.70), we obtain

$$Q_{mk} < \frac{|\alpha_m^{-1}| - \alpha}{|\alpha_k| - \alpha}. \tag{5.72}$$

For $m \le k$ the estimation (5.72) is no more than (5.71). In fact, from (5.68) for $m \le k$ we have $|\alpha_m| \ge |\alpha_k| \ge 1$. Therefore,

$$\frac{|\alpha_m^{-1}| - \alpha}{|\alpha_k| - \alpha} \le \frac{|\alpha_k^{-1}| - \alpha}{|\alpha_k| - \alpha} \le \frac{1 - \alpha^2}{\alpha_k^2 - \alpha^2}.$$

To get a stability condition for (5.60) consider again the functional $V_{1i} = \zeta_i x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ via (5.60) we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[\zeta_{i+1}(\eta_i x_i + b x_{i-h} + \sigma x_{i-l}\xi_{i+1})^2 - \zeta_i x_i^2\big]\\ &= \gamma_i \mathbf{E}x_i^2 + \mathbf{E}\zeta_{i+1}\big(b^2 x_{i-h}^2 + \sigma^2 x_{i-l}^2 + 2b\eta_i x_i x_{i-h}\big)\\ &\le \gamma_i \mathbf{E}x_i^2 + \mu_i\big(b^2\mathbf{E}x_{i-h}^2 + \sigma^2\mathbf{E}x_{i-l}^2\big) + \rho_i|b|\big(\mathbf{E}x_i^2 + \mathbf{E}x_{i-h}^2\big)\\ &= \big(\gamma_i + \rho_i|b|\big)\mathbf{E}x_i^2 + \big(\mu_i b^2 + \rho_i|b|\big)\mathbf{E}x_{i-h}^2 + \mu_i\sigma^2\mathbf{E}x_{i-l}^2,\end{aligned}$$

where γ_i is defined by (5.64), $\rho_i = \mathbf{E}\zeta_{i+1}|\eta_i|$, $\mu_i = \mathbf{E}\zeta_{i+1}$. Put $\gamma = \sup_{i\in Z}\gamma_i$, $\rho = \sup_{i\in Z}\rho_i$, $\mu = \sup_{i\in Z}\mu_i$. Via Theorem 1.2 the inequality

$$\gamma + 2\rho|b| + \mu\big(b^2 + \sigma^2\big) < 0 \tag{5.73}$$

is the sufficient condition for asymptotic mean square stability of the trivial solution of (5.60).

Consider condition (5.73) in more detail. Suppose that (5.68) holds and $\zeta_i \le \beta_1$. Then $\rho \le |\alpha_1|\beta_1$, $\mu \le \beta_1$ and $2\rho|b| + \mu(b^2+\sigma^2) \le A\beta_1$, where $A = 2|\alpha_1 b| + b^2 + \sigma^2$. As a result we obtain the following.

Lemma 5.3 *Let $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ be the states of the Markov chains η_i and ζ_i with the probabilities of transition* (5.61) *and* (5.63). *If*

$$\begin{gathered}\sup_{m\ge 1}\left\{\alpha_m^2\sum_{l=1}^{\infty}P_{ml}\beta_l - \beta_m\right\} + A\beta_1 < 0,\\ A = 2|\alpha_1 b| + b^2 + \sigma^2,\end{gathered} \tag{5.74}$$

then the trivial solution of (5.60) *is asymptotically mean square stable.*

Example 5.4 Let us suppose that the Markov chain η_i in the difference (5.62) has two states α_1, α_2, such that $|\alpha_1| \ge |\alpha_2| > 0$. If $|\alpha_1| < 1$ then (Remark 5.1) the trivial solution of (5.62) is asymptotically mean square stable. If $|\alpha_2| \ge 1$ then (Remark 5.2) the trivial solution of (5.62) cannot be asymptotically mean square stable.

Suppose that

$$|\alpha_2| < 1 \le |\alpha_1|. \tag{5.75}$$

From (5.71) it follows that for asymptotic mean square stability the probability P_{11} must be small enough, i.e.

$$P_{11} < \frac{1-\alpha_2^2}{\alpha_1^2 - \alpha_2^2}.$$

If besides of (5.75) $|\alpha_1\alpha_2| < 1$, then via (5.72) another estimation, namely

$$P_{11} < \frac{|\alpha_1^{-1}| - |\alpha_2|}{|\alpha_1| - |\alpha_2|},$$

holds that is no more than the previous one.

Let us obtain a condition for the asymptotic mean square stability of the trivial solution of (5.60) in terms of the parameters α_m, P_{mm}, $m = 1, 2$, b, σ^2. Via (5.74) we have

$$\alpha_m^2(P_{m1}\beta_1 + P_{m2}\beta_2) + A\beta_1 < \beta_m, \quad m = 1, 2. \tag{5.76}$$

Put $\lambda = \beta_1^{-1}\beta_2$ and rewrite (5.76) as follows:

$$\alpha_1^2(P_{11} + P_{12}\lambda) + A < 1, \qquad \alpha_2^2(P_{21} + P_{22}\lambda) + A < \lambda.$$

From this we have

$$\frac{1 + \alpha_2^{-2}A - P_{22}}{\alpha_2^{-2} - P_{22}} < \lambda < \frac{\alpha_1^{-2}(1 - A) - P_{11}}{1 - P_{11}}. \tag{5.77}$$

It means that if

$$\frac{1 + \alpha_2^{-2}A - P_{22}}{\alpha_2^{-2} - P_{22}} < \frac{\alpha_1^{-2}(1 - A) - P_{11}}{1 - P_{11}} \tag{5.78}$$

then there exist β_1, β_2, such that $0 < \beta_2 < \beta_1$ (i.e. $0 < \lambda < 1$), for which condition (5.77) holds. Using the representation for A, transform the condition (5.78) into the form

$$A = 2|\alpha_1 b| + b^2 + \sigma^2,$$

$$\alpha_1^2 < \frac{(\alpha_2^{-2} - P_{22})(1 - A)}{1 - P_{22} + (\alpha_2^{-2} - 1)P_{11} + A\alpha_2^{-2}(1 - P_{11})}. \tag{5.79}$$

So, if condition (5.79) holds then the trivial solution of (5.60) is asymptotically mean square stable.

If $b = \sigma = 0$ then condition (5.79) takes the form

$$\alpha_1^2 < \frac{\alpha_2^{-2} - P_{22}}{1 - P_{22} + (\alpha_2^{-2} - 1)P_{11}} \tag{5.80}$$

and is the sufficient condition for asymptotic mean square stability of the trivial solution of (5.62).

In Fig. 5.13 stability regions for the trivial solution of (5.60) obtained by the condition (5.79) are shown in the plane (λ_1, λ_2), where $\lambda_i = |\alpha_i|$, $i = 1, 2$, for $P_{11} = 0.05$, $\sigma^2 = 0.01$, $b = 0$ and different values of P_{22}: (1) $P_{22} = 0$, (2) $P_{22} = 0.8$, (3) $P_{22} = 0.95$, (4) $P_{22} = 1$. In Fig. 5.14 similar stability regions are shown for $P_{11} = 0.05$, $\sigma^2 = 0$, $b = 0.01$.

Stability regions for the trivial solution of (5.62) obtained by the condition (5.80) are shown for the same values of P_{22} and for $P_{11} = 0.05$ in Fig. 5.15 and for $P_{11} = 0.1$ in Fig. 5.16. In Figs. 5.15 and 5.16 the parts of the region marked by the letter M satisfy the condition (5.80), the parts of the region marked by the letter N satisfy the

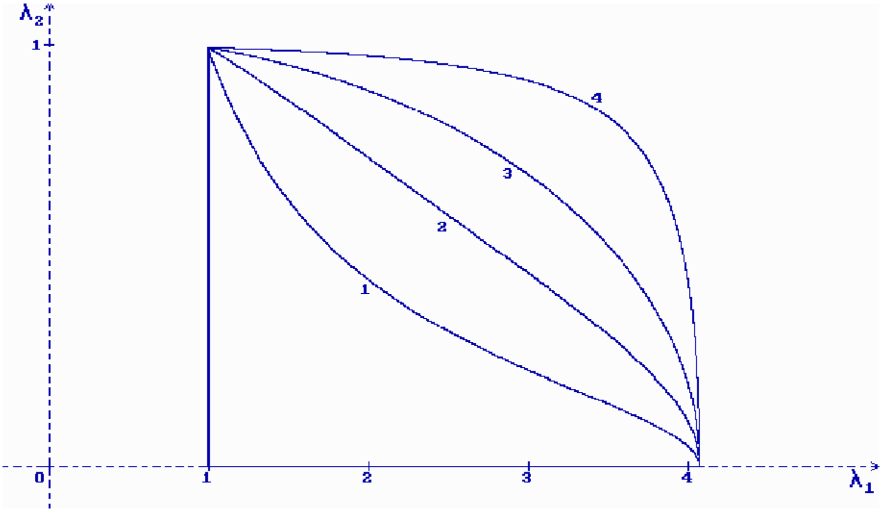


Fig. 5.13 Stability regions for (5.60) given by condition (5.79) for $P_{11} = 0.05$, $\sigma^2 = 0.01$, $b = 0$ and different values of P_{22}: (*1*) $P_{22} = 0$, (*2*) $P_{22} = 0.8$, (*3*) $P_{22} = 0.95$, (*4*) $P_{22} = 1$

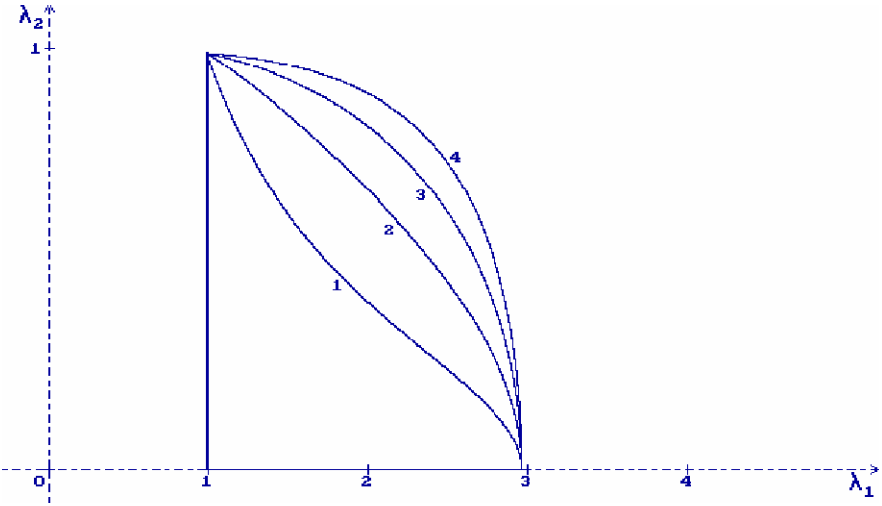


Fig. 5.14 Stability regions for (5.60) given by the condition (5.79) for $P_{11} = 0.05$, $\sigma^2 = 0$, $b = 0.01$ and different values of P_{22}: (*1*) $P_{22} = 0$, (*2*) $P_{22} = 0.8$, (*3*) $P_{22} = 0.95$, (*4*) $P_{22} = 1$

condition $|\alpha_2| \le |\alpha_1| < 1$. As follows from Remark 5.1, by this condition the trivial solution of (5.62) is asymptotically mean square stable for arbitrary P_{11} and P_{22}.

Remark 5.5 Note that if $P_{22} = 1$, i.e. the set α_2 of the Markov chain η_i is an absorbed set, then from (5.80) it follows that the trivial solution of (5.62) is asymptotically mean square stable if $\alpha_1^2 < P_{11}^{-1}$.

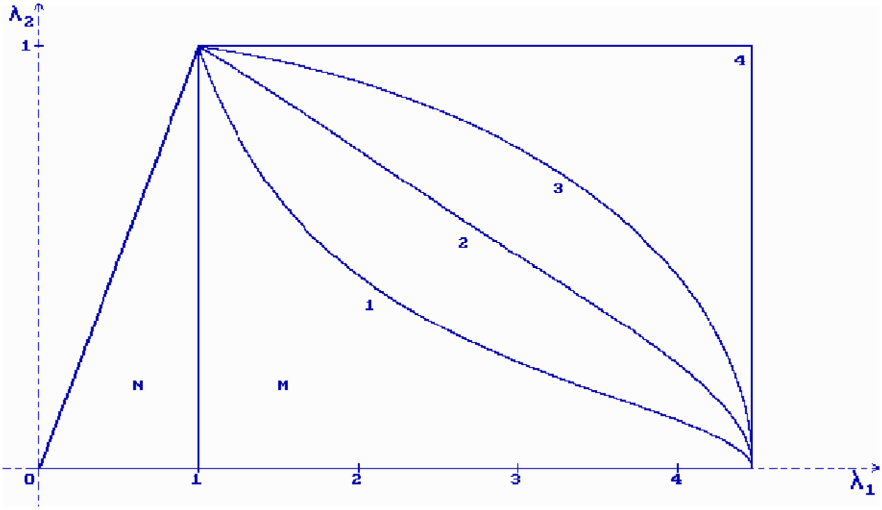


Fig. 5.15 Stability regions for (5.62) given by the condition (5.80) for $P_{11} = 0.05$ and different values of P_{22}: (*1*) $P_{22} = 0$, (*2*) $P_{22} = 0.8$, (*3*) $P_{22} = 0.95$, (*4*) $P_{22} = 1$

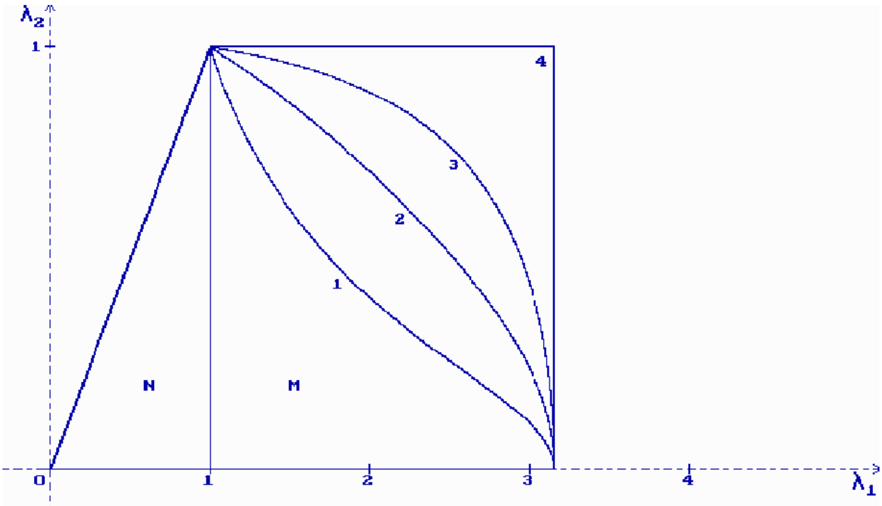


Fig. 5.16 Stability regions for (5.62) given by the condition (5.80) for $P_{11} = 0.1$ and different values of P_{22}: (*1*) $P_{22} = 0$, (*2*) $P_{22} = 0.8$, (*3*) $P_{22} = 0.95$, (*4*) $P_{22} = 1$

Suppose now that the Markov chain η_i in (5.60) has two states α_1, α_2 with condition (5.73). Let us suppose also that

$$P_{11} = 0, \qquad P_{22} = 1. \tag{5.81}$$

It means that α_1 is the unessential state and α_2 is the absorbent state; then from (5.79) and (5.81) it follows that

$$2|\alpha_1 b| + b^2 + \sigma^2 < \frac{1-\alpha_2^2}{1-\alpha_2^2+\alpha_1^2} < 1. \tag{5.82}$$

Let us show that the condition (5.82) can be essentially relaxed. In fact, it is easy to see that by the conditions (5.81) we have $P\{\eta_i = \alpha_1\} = 0$, $P\{\eta_i = \alpha_2\} = 1$, $i > 0$. Therefore, for $i > 0$ we obtain $\gamma_i = (\alpha_2^2 - 1)\beta_2$, $\rho_i = |\alpha_2|\beta_2$, $\mu_i = \beta_2$, where $\beta_2 > 0$. So the condition (5.73) takes the form

$$\big(|\alpha_2| + |b|\big)^2 + \sigma^2 < 1. \tag{5.83}$$

Thus, if the Markov chain η_i in (5.60) has two states α_1, α_2 with conditions (5.75) and (5.81), then the inequality (5.83) is the sufficient condition for asymptotic mean square stability of the trivial solution of (5.60).

It is easy to see also that the condition (5.83) coincides with (2.2).

Chapter 6
Systems of Linear Equations with Varying Delays

Here the general method of the construction of Lyapunov functionals is used for constructing of asymptotic mean square stability conditions for systems of stochastic linear difference equations with varying delays. Stability conditions are formulated in terms of the existence of positive definite solutions of certain matrix Riccati equations.

6.1 Systems with Nonincreasing Delays

Consider the system of the stochastic linear difference equations

$$x_{i+1} = Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_{i+1}. \tag{6.1}$$

Here A, B, C are square n-dimensional matrices, $x_i \in R^n$, $h = \max(k(0), m(0))$ and the delays $k(i)$ and $m(i)$ satisfy the inequalities

$$k(i) \geq k(i+1) \geq 0, \qquad m(i) \geq m(i+1) \geq 0, \quad i \in Z. \tag{6.2}$$

Below, two ways of the construction of the Lyapunov functionals for (6.1) are considered.

6.1.1 First Way of the Construction of the Lyapunov Functional

Using representation (1.7) let us put (Step 1) $\tau = 0$:

$$F_1(i, x_i) = Ax_i, \qquad F_2(i, x_{-h}, \dots, x_i) = Bx_{i-k(i)},$$

$$F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i,$$

DOI 10.1007/978-0-85729-685-6_6,

$$G_2(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i-1,$$
$$G_2(i, i, x_{-h}, \dots, x_i) = Cx_{i-m(i)}, \quad i \in Z.$$

In this case the auxiliary equation (Step 2) has the form

$$y_{i+1} = Ay_i. \tag{6.3}$$

Let for some positive definite matrix Q_0 the matrix equation

$$A'DA - D = -Q_0 \tag{6.4}$$

have a positive definite solution D. Then the function $v_i = y_i'Dy_i$ is a Lyapunov function for (6.3). Actually, calculating Δv_i and using (6.3) and (6.4), we get

$$\Delta v_i = y_{i+1}'Dy_{i+1} - y_i'Dy_i = y_i'A'DAy_i - y_i'Dy_i = -y_i'Q_0y_i.$$

We will construct (Step 3) the Lyapunov functional V_i for (6.1) in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i'Dx_i$. Calculating $\mathbf{E}\Delta V_{1i}$ for (6.1) we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}[x_{i+1}'Dx_{i+1} - x_i'Dx_i] \\
&= \mathbf{E}\big[(Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_i)'D\big(Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_i\big) \\
&\quad - x_i'Dx_i\big] \\
&= \mathbf{E}\big[x_i'\big(A'DA - D\big)x_i + x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i \\
&\quad + x_{i-k(i)}'B'DBx_{i-k(i)} + x_{i-m(i)}'C'DCx_{i-m(i)}\big].
\end{aligned}$$

Via Lemma 1.3 for $a = Ax_i$, $b = DBx_{i-k(i)}$ and an arbitrary positive definite matrix R we have

$$x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i \le x_i'A'RAx_i + x_{i-k(i)}'B'DR^{-1}DBx_{i-k(i)}. \tag{6.5}$$

Therefore

$$\mathbf{E}\Delta V_{1i} \le \mathbf{E}\big[x_i'\big(A'DA - D + A'RA\big)x_i + u_{i-k(i)} + z_{i-m(i)}\big], \tag{6.6}$$

where

$$u_i = x_i'B'\big(DR^{-1}D + D\big)Bx_i, \qquad z_i = x_i'C'DCx_i.$$

Choose the additional functional V_{2i} (Step 4) in the form

$$V_{2i} = \sum_{j=i-k(i)}^{i-1} u_j + \sum_{j=i-m(i)}^{i-1} z_j.$$

Then using (6.2) we obtain

$$\begin{aligned}\Delta V_{2i} &= \sum_{j=i+1-k(i+1)}^{i} u_j + \sum_{j=i+1-m(i+1)}^{i} z_j - V_{2i}\\ &= u_i + z_i + \sum_{j=i+1-k(i+1)}^{i-1} u_j + \sum_{j=i+1-m(i+1)}^{i-1} z_j\\ &\quad - \sum_{j=i+1-k(i)}^{i-1} u_j - \sum_{j=i+1-m(i)}^{i-1} z_j - u_{i-k(i)} - z_{i-m(i)}\\ &\le u_i + z_i - u_{i-k(i)} - z_{i-m(i)}.\end{aligned}$$

From this and (6.6) for the functional $V_i = V_{1i} + V_{2i}$ it follows

$$\mathbf{E}\Delta V_i \le -\mathbf{E}x_i' Q x_i, \tag{6.7}$$

where

$$-Q = A'DA - D + B'DB + C'DC + A'RA + B'DR^{-1}DB. \tag{6.8}$$

Remark 6.1 Using Lemma 1.3 for $a = Bx_{i-k(i)}$ and $b = DAx_i$, instead of (6.5) we obtain the inequality

$$x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i \le x_i'A'DR^{-1}DAx_i + x_{i-k(i)}'B'RBx_{i-k(i)}.$$

As a result instead of matrix equation (6.8) we obtain

$$-Q = A'DA - D + B'DB + C'DC + B'RB + A'DR^{-1}DA. \tag{6.9}$$

Using Lemma 1.3 for $a = x_i$, $b = A'DBx_{i-k(i)}$ instead of (6.8) we obtain

$$-Q = A'DA - D + B'DB + C'DC + R + B'DAR^{-1}A'DB. \tag{6.10}$$

Using Lemma 1.3 for $a = x_{i-k(i)}$, $b = B'DAx_i$ instead of (6.8) we obtain

$$-Q = A'DA - D + B'DB + C'DC + R + A'DBR^{-1}B'DA. \tag{6.11}$$

So, we get the following.

Theorem 6.1 *Let for some positive definite matrices Q and R a positive definite solution D exist of at least one of the matrix Riccati equations* (6.8), (6.9), (6.10) *or* (6.11). *Then the trivial solution of* (6.1) *is asymptotically mean square stable.*

Note that using different matrices R and Q and different equations type of (6.8)–(6.11) *one can obtain different delay-independent stability conditions.*

Example 6.1 In the scalar case the positive solution of (6.8)–(6.11) exists if and only if

$$\big(|A| + |B|\big)^2 + C^2 < 1. \tag{6.12}$$

Example 6.2 Consider the two-dimensional system

$$\begin{aligned} x_{i+1} &= a_1 x_i + b y_{i-k(i)} + c_1 x_{i-m(i)} \xi_{i+1}, \\ y_{i+1} &= a_2 y_i + c_2 y_{i-m(i)} \xi_{i+1}. \end{aligned} \tag{6.13}$$

Putting $R = I$, $Q = qI$, where $q > 0$, I is identity matrix, we see that the solution D of (6.11) has the elements $d_{12} = 0$,

$$d_{11} = \frac{1 - a_1^2 - c_1^2 + \sqrt{(1 - a_1^2 - c_1^2)^2 - 4(1+q)b^2 a_1^2}}{2b^2 a_1^2}, \qquad d_{22} = \frac{b^2 d_{11} + 1 + q}{1 - a_2^2 - c_2^2}.$$

Using small enough $q > 0$ we obtain the sufficient condition for asymptotic mean square stability of the trivial solution of system (6.13) in the form

$$a_1^2 + c_1^2 + 2|ba_1| < 1, \qquad a_2^2 + c_2^2 < 1. \tag{6.14}$$

On the other hand the solution D of (6.8) has the elements $d_{12} = 0$,

$$d_{11} = \frac{a_1^2 + q}{1 - a_1^2 - c_1^2}, \qquad d_{22} = \frac{b^2 d_{11}^2 + b^2 d_{11} + a_2^2 + q}{1 - a_2^2 - c_2^2}.$$

So we obtain another sufficient condition for asymptotic mean square stability of the trivial solution of system (6.13), which is weaker than (6.14): $a_i^2 + c_i^2 < 1$, $i = 1, 2$.

6.1.2 Second Way of the Construction of the Lyapunov Functional

Let us use now take the representation (1.7) (Step 1) for $\tau = 0$,

$$\begin{gathered} F_1(i) = F_1(i, x_i) = (A + B)x_i, \\ F_2(i) = F_2(i, x_{-h}, \dots, x_i) = -\sum_{j=i+1-k(i)}^{i-k(i+1)} B x_j, \\ F_3(i) = F_3(i, x_{-h}, \dots, x_i) = -\sum_{j=i-k(i)}^{i-1} B x_j, \\ G_1(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i, \\ G_2(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i-1, \\ G_2(i) = G_2(i, i, x_{-h}, \dots, x_i) = C x_{i-m(i)}. \end{gathered}$$

In this case the auxiliary equation (Step 2) has the form

$$y_{i+1} = (A + B) y_i. \tag{6.15}$$

Let for some positive definite matrix Q_0 the matrix equation

$$(A+B)'D(A+B)-D=-Q_0$$

have a positive definite solution D. Then the function $v_i=y_i'Dy_i$ is a Lyapunov function for (6.15).

We will construct the Lyapunov functional V_i for (6.1) in the form $V_i=V_{1i}+V_{2i}$, where (Step 3) $V_{1i}=(x_i-F_3(i))'D(x_i-F_3(i))$. Calculating $\mathbf{E}\Delta V_{1i}$ we get

$$\begin{aligned}\mathbf{E}\Delta V_{1i}&=\mathbf{E}\big[\big(x_{i+1}-F_3(i+1)\big)'D\big(x_{i+1}-F_3(i+1)\big)-V_{1i}\big]\\&=\mathbf{E}\big(F_1(i)-x_i+F_2(i)+G_2(i)\xi_{i+1}\big)'D\big(F_1(i)+x_i\\&\quad+F_2(i)-2F_3(i)+G_2(i)\xi_{i+1}\big)\\&=\mathbf{E}\left[x_i'\big((A+B)'D(A+B)-D\big)x_i+\sum_{j=1}^{5}I_j\right],\end{aligned}$$

where

$$\begin{gathered}I_1=2F_1'(i)DF_2(i),\qquad I_2=F_2'(i)DF_2(i),\\I_3=-2\big(F_1(i)-x_i\big)'DF_3(i),\\I_4=2F_2'(i)DF_3(i),\qquad I_5=G_2'(i)DG_2(i).\end{gathered}$$

Put

$$k_0=\sup_{i\in Z}\big(k(i)-k(i+1)\big). \tag{6.16}$$

Using (6.16) and Lemma 1.3 for $a=DBx_j$, $b=(A+B)x_i$ and a positive definite matrix R, we obtain

$$\begin{aligned}I_1&=-\sum_{j=i+1-k(i)}^{i-k(i+1)}\big(x_j'B'D(A+B)x_i+x_i'(A+B)'DBx_j\big)\\&\le\sum_{j=i+1-k(i)}^{i-k(i+1)}\big(x_j'B'DRDBx_j+x_i'(A+B)'R^{-1}(A+B)x_i\big)\\&\le k_0x_i'(A+B)'R^{-1}(A+B)x_i+\sum_{j=i+1-k(i)}^{i-k(i+1)}x_j'B'DRDBx_j.\end{aligned}$$

It is easy to see that

$$I_2 = \sum_{l,j=i+1-k(i)}^{i-k(i+1)} x'_j B' D B x_l = \left| \sum_{j=i+1-k(i)}^{i-k(i+1)} D^{\frac{1}{2}} B x_j \right|^2$$

$$\leq k_0 \sum_{j=i+1-k(i)}^{i-k(i+1)} \left| D^{\frac{1}{2}} B x_j \right|^2 = k_0 \sum_{j=i+1-k(i)}^{i-k(i+1)} x'_j B' D B x_j.$$

Using Lemma 1.3 for $a = DBx_j$, $b = (A + B - I)x_i$ and a positive definite matrix P we get

$$I_3 = - \sum_{j=i-k(i)}^{i-1} \big(x'_j B' D (A + B - I) x_i + x'_i (A + B - I)' D B x_j \big)$$

$$\leq \sum_{j=i-k(i)}^{i-1} \big(x'_j B' D P D B x_j + x'_i (A + B - I)' P^{-1} (A + B - I) x_i \big)$$

$$\leq k(0) x'_i (A + B - I)' P^{-1} (A + B - I) x_i + \sum_{j=i-k(i)}^{i-1} x'_j B' D P D B x_j.$$

Using Lemma 1.3 for $a = Bx_j$, $b = DBx_i$ and a positive definite matrix S we get

$$I_4 = - \sum_{l=i-k(i)}^{i-1} \sum_{j=i+1-k(i)}^{i-k(i+1)} \big(x'_j B' D B x_l + x'_l B' D B x_j \big)$$

$$\leq \sum_{l=i-k(i)}^{i-1} \sum_{j=i+1-k(i)}^{i-k(i+1)} \big(x'_j B' S B x_j + x'_l B' D S^{-1} D B x_l \big)$$

$$\leq k(0) \sum_{j=i+1-k(i)}^{i-k(i+1)} x'_j B' S B x_j + k_0 \sum_{l=i-k(i)}^{i-1} x'_l B' D S^{-1} D B x_l.$$

Let $q_i = x'_i C' D C x_i$. Then $I_5 = q_{i-m(i)}$. As a result we have

$$\begin{aligned} \mathbf{E}\Delta V_{1i} \leq \mathbf{E}\big[x'_i \big(&(A + B)' D (A + B) - D \\ &+ k_0 (A + B)' R^{-1} (A + B) + k(0) (A + B - I)' P^{-1} (A + B - I) \big) x_i \big] \\ &+ \mathbf{E}\left[\sum_{j=i+1-k(i)}^{i-k(i+1)} u_j + \sum_{j=i-k(i)}^{i-1} z_j + q_{i-m(i)} \right], \end{aligned}$$

where

$$u_j = x'_j \big(B' D R D B + k_0 B' D B + k(0) B' S B \big) x_j,$$

$$z_j = x'_j \big(B' D P D B + k_0 B' D S^{-1} D B \big) x_j.$$

Put $k_m = \inf_{i \in Z} k(i)$ and choose the additional functional V_{2i} (Step 4) in the form

$$V_{2i} = \sum_{l=i}^{i+k_m-2} \sum_{j=l+1-k(0)}^{l-k_m} u_j + \sum_{j=i+k_m-k(0)}^{i-1} \big(j - i - k_m + k(0) + 1\big) u_j + \sum_{j=i-k(0)}^{i-1} \big(j - i + k(0) + 1\big) z_j + \sum_{j=i-m(i)}^{i-1} q_j.$$

Calculating ΔV_{2i}, we have

$$\begin{aligned}\Delta V_{2i} &= \sum_{l=i+1}^{i+k_m-1} \sum_{j=l+1-k(0)}^{l-k_m} u_j + \sum_{j=i+1+k_m-k(0)}^{i} \big(j - i - k_m + k(0)\big) u_j \\ &\quad + \sum_{j=i+1-k(0)}^{i} \big(j - i + k(0)\big) z_j + \sum_{j=i+1-m(i+1)}^{i} q_j - V_{2i} \\ &= \big(k(0) - k_m\big) u_i + k(0) z_i + q_i - q_{i-m(i)} \\ &\quad - \sum_{j=i+1-k(0)}^{i-k_m} u_j - \sum_{j=i-k(0)}^{i-1} z_j + \sum_{j=i+1-m(i+1)}^{i-1} q_j - \sum_{j=i+1-m(i)}^{i-1} q_j.\end{aligned}$$

Using that $k(0) \geq k(i) \geq k_m$ and $m(i) \geq m(i+1)$ for the functional $V_i = V_{1i} + V_{2i}$ we obtain the inequality (6.7) with

$$\begin{aligned}-Q &= (A+B)' D (A+B) - D + k_0 (A+B)' R^{-1} (A+B) \\ &\quad + k(0)(A+B-I)' P^{-1} (A+B-I) + C' D C \\ &\quad + \big(k(0) - k_m\big) B' \big(DRD + k_0 D + k(0) S\big) B \\ &\quad + k(0) B' D \big(P + k_0 S^{-1}\big) D B.\end{aligned} \tag{6.17}$$

Theorem 6.2 *Let for some positive definite matrices Q, P, R and S the positive definite solution D exist of the matrix Riccati equation* (6.17). *Then the trivial solution of* (6.1) *is asymptotically mean square stable.*

Remark 6.2 Similar to Remark 6.1 one can show that instead of (6.17) in Theorem 6.2 there can be used other matrix Riccati equations, for example

$$\begin{aligned}-Q &= (A+B)' D (A+B) - D + k_0 (A+B)' D R^{-1} D (A+B) \\ &\quad + k(0)(A+B-I)' D P^{-1} D (A+B-I) + C' D C \\ &\quad + \big(k(0) - k_m\big) B' \big(R + k_0 D + k(0) S\big) B \\ &\quad + k(0) B' \big(P + k_0 D S^{-1} D\big) B.\end{aligned} \tag{6.18}$$

Remark 6.3 Let $k(i) = k = \text{const}$. In this case (6.17) and (6.18), respectively, are

$$\begin{aligned}
-Q &= (A+B)'D(A+B) - D + C'DC \\
&\quad + k(A+B-I)'P^{-1}(A+B-I) + kB'DPDB, \\
-Q &= (A+B)'D(A+B) - D + C'DC \\
&\quad + k(A+B-I)'DP^{-1}D(A+B-I) + kB'PB.
\end{aligned}$$

Example 6.3 It is easy to see that in the scalar case a positive solution of (6.17) (or (6.18)) exists if and only if

$$\begin{aligned}
&\big(|A+B| + \gamma|B|\big)^2 + 2k(0)|B|\big(1 - A - B + \gamma|B|\big) + C^2 < 1, \\
&|A+B| < 1, \ \gamma = \sqrt{k_0(k(0) - k_m)}.
\end{aligned}$$

If $k(i) = k = \text{const}$ then $\gamma = 0$, and this condition can be written in the form

$$C^2 < (1 - A - B)\big(1 + A + B - 2k|B|\big), \quad |A+B| < 1. \tag{6.19}$$

6.2 Systems with Unbounded Delays

Construct now the asymptotic mean square stability conditions for the stochastic linear difference equation

$$x_{i+1} = \sum_{j=0}^{k(i)} \alpha_j A_j x_{i-j} + \sum_{j=0}^{m(i)} \beta_j B_j x_{i-j} \xi_{i+1}. \tag{6.20}$$

Here A_j and B_j are $n \times n$-matrices, α_j and β_j are scalars. It is supposed that the delays $k(i)$ and $m(i)$ satisfy the inequalities

$$k(i+1) - k(i) \le 1, \qquad m(i+1) - m(i) \le 1 \tag{6.21}$$

and $h = \max(\hat{k}, \hat{m})$, where

$$\hat{k} = \sup_{i \in Z} k(i) \le \infty, \qquad \hat{m} = \sup_{i \in Z} m(i) \le \infty. \tag{6.22}$$

6.2.1 *First Way of the Construction of the Lyapunov Functional*

Let us represent (Step 1) (6.20) in the form (1.7) by $\tau = 0$,

$$F_1(i, x_i) = \alpha_0 A_0 x_i, \qquad F_2(i, x_{-h}, \dots, x_i) = \sum_{j=1}^{k(i)} \alpha_j A_j x_{i-j},$$

$$F_3(i, x_{-h}, \dots, x_i) = 0,$$

$$G_1(i, j, x_{-h}, \dots, x_j) = G_2(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i-1,$$

$$G_1(i, i, x_{-h}, \dots, x_i) = \beta_0 B_0 x_i, \qquad G_2(i, i, x_{-h}, \dots, x_i) = \sum_{j=1}^{m(i)} \beta_j B_j x_{i-j}.$$

In this case the auxiliary equation (Step 2) has the form

$$y_{i+1} = \alpha_0 A_0 y_i + \beta_0 B_0 y_i \xi_{i+1}.$$

Let for some positive definite matrix Q_0 the matrix equation

$$\alpha_0^2 A_0' D A_0 + \beta_0^2 B_0' D B_0 - D = -Q_0$$

have a positive definite solution D. Then the function $v_i = y_i' D y_i$ is a Lyapunov function for the auxiliary equation. Actually, calculating $\mathbf{E}\Delta v_i$ we have

$$\begin{aligned}
\mathbf{E}\Delta v_i &= \mathbf{E}(y_{i+1}' D y_{i+1} - y_i' D y_i) \\
&= \mathbf{E}\big[(\alpha_0 A_0 y_i + \beta_0 B_0 y_i \xi_{i+1})' D (\alpha_0 A_0 y_i + \beta_0 B_0 y_i \xi_{i+1}) - y_i' D y_i\big] \\
&= \mathbf{E}\big[\alpha_0^2 y_i' A_0' D A_0 y_i + \beta_0^2 y_i' B_0' D B_0 y_i - y_i' D y_i\big] = -\mathbf{E} y_i' Q_0 y_i.
\end{aligned}$$

We will construct a Lyapunov functional for (6.16) in the form $V_i = V_{1i} + V_{2i}$, where (Step 3) $V_{1i} = x_i' D x_i$. Let

$$\alpha = \sum_{l=0}^{\hat{k}} |\alpha_l|, \qquad \beta = \sum_{l=0}^{\hat{m}} |\beta_l|.$$

Calculating $\mathbf{E}\Delta V_{1i}$ for (6.20) we get

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\Bigg[\Bigg(\sum_{j=0}^{k(i)} \alpha_j A_j x_{i-j} + \sum_{j=0}^{m(i)} \beta_j B_j x_{i-j} \xi_{i+1}\Bigg)' D \Bigg(\sum_{j=0}^{k(i)} \alpha_j A_j x_{i-j} \\
&\qquad + \sum_{j=0}^{m(i)} \beta_j B_j x_{i-j} \xi_{i+1}\Bigg) - x_i' D x_i\Bigg] \\
&= \mathbf{E}\Bigg[\Bigg|\sum_{j=0}^{k(i)} \alpha_j D^{\frac{1}{2}} A_j x_{i-j}\Bigg|^2 + \Bigg|\sum_{j=0}^{m(i)} \beta_j D^{\frac{1}{2}} B_j x_{i-j}\Bigg|^2 - x_i' D x_i\Bigg] \\
&\le \mathbf{E}\Bigg[\sum_{l=0}^{k(i)} |\alpha_l| \sum_{j=0}^{k(i)} |\alpha_j| \big|D^{\frac{1}{2}} A_j x_{i-j}\big|^2
\end{aligned}$$

$$+\sum_{l=0}^{m(i)}|\beta_l|\sum_{j=0}^{m(i)}|\beta_j|\left|D^{\frac{1}{2}}B_jx_{i-j}\right|^2-x_i'Dx_i\Bigg]$$

$$\le \mathbf{E}\big[x_i'(K_0+L_0-D)x_i+J_{1i}+J_{2i}\big],$$

where

$$J_{1i}=\sum_{j=1}^{k(i)}x_{i-j}'K_jx_{i-j},\qquad J_{2i}=\sum_{j=1}^{m(i)}x_{i-j}'L_jx_{i-j},$$
$$K_j=\alpha|\alpha_j|A_j'DA_j,\qquad L_j=\beta|\beta_j|B_j'DB_j. \tag{6.23}$$

Let us choose the additional functional V_{2i} (Step 4) in the form $V_{2i}=V_{2i}^{(1)}+V_{2i}^{(2)}$, where

$$V_{2i}^{(1)}=\sum_{j=1}^{k(i)}x_{i-j}'Z_j^{(1)}x_{i-j},\quad Z_j^{(1)}=\sum_{l=j}^{\hat{k}}K_l, \tag{6.24}$$

$$V_{2i}^{(2)}=\sum_{j=1}^{m(i)}x_{i-j}'Z_j^{(2)}x_{i-j},\quad Z_j^{(2)}=\sum_{l=j}^{\hat{m}}L_l. \tag{6.25}$$

Then using (6.21) we get

$$\begin{aligned}\Delta V_{2i}^{(1)}&=\sum_{j=1}^{k(i+1)}x_{i+1-j}'Z_j^{(1)}x_{i+1-j}-\sum_{j=1}^{k(i)}x_{i-j}'Z_j^{(1)}x_{i-j}\\&=\sum_{j=0}^{k(i+1)-1}x_{i-j}'Z_{j+1}^{(1)}x_{i-j}-\sum_{j=1}^{k(i)}x_{i-j}'Z_{j+1}^{(1)}x_{i-j}-J_{1i}\\&=x_i'Z_1^{(1)}x_i-J_{1i}+\sum_{j=1}^{k(i+1)-1}x_{i-j}'Z_{j+1}^{(1)}x_{i-j}-\sum_{j=1}^{k(i)}x_{i-j}'Z_{j+1}^{(1)}x_{i-j}\\&\le x_i'Z_1^{(1)}x_i-J_{1i}.\end{aligned} \tag{6.26}$$

Analogously

$$\Delta V_{2i}^{(2)}\le x_i'Z_1^{(2)}x_i-J_{2i}. \tag{6.27}$$

As a result for the functional V_i we have the inequality (6.7), where

$$-Q=\alpha\sum_{j=1}^{\hat{k}}|\alpha_j|A_j'DA_j+\beta\sum_{j=1}^{\hat{m}}|\beta_j|B_j'DB_j-D. \tag{6.28}$$

Theorem 6.3 *Let for some positive definite matrix Q the positive definite solution of matrix equation* (6.28) *exist. Then the trivial solution of* (6.20) *is asymptotically mean square stable.*

Remark 6.4 Let $A_j = A = \text{const}$, $B_j = B = \text{const}$. Then the matrix equation (6.28) takes the form

$$-Q = \alpha^2 A' DA + \beta^2 B' DB - D.$$

If $A = B = I$ then the asymptotic mean square stability condition has the form

$$\alpha^2 + \beta^2 < 1. \tag{6.29}$$

It is easy to see that the condition (6.12) is a particular case of condition (6.29).

Remark 6.5 From the inequalities (6.21) it follows that

$$k(i) \le k(0) + i, \qquad m(i) \le m(0) + i.$$

6.2.2 Second Way of the Construction of the Lyapunov Functional

Consider now another way of the construction of the Lyapunov functionals for the stochastic difference equation

$$x_{i+1} = \sum_{j=0}^{k(i)} A_j x_{i-j} + \sum_{j=0}^{m(i)} B_j x_{i-j} \xi_{i+1}, \tag{6.30}$$

where the delays $k(i)$ and $m(i)$ satisfy the conditions (6.21) and (6.22).

Let us represent (6.30) (Step 1) in the form (1.7), where $\tau = 0$,

$$F_1(i, x_i) = W_0 x_i, \qquad F_2(i) = F_2(i, x_{-h}, \dots, x_i) = -\sum_{j=k(i+1)}^{k(i)} W_{j+1} x_{i-j},$$

$$F_3(i) = F_3(i, x_{-h}, \dots, x_i) = -\sum_{j=1}^{k(i)} W_j x_{i-j}, \quad W_l = \sum_{j=l}^{\hat{k}} A_j,$$

$$G_1(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i,$$

$$G_2(i, j, x_{-h}, \dots, x_j) = 0, \quad j = 0, \dots, i-1,$$

$$G_2(i) = G_2(i, i, x_{-h}, \dots, x_i) = \sum_{j=0}^{m(i)} B_j x_{i-j}.$$

Actually, it is easy to see that

$$\begin{aligned}
&F_1(i,x_i)+F_2(i)+\Delta F_3(i)\\
&=W_0x_i-\sum_{j=k(i+1)}^{k(i)}W_{j+1}x_{i-j}-\sum_{j=1}^{k(i+1)}W_jx_{i+1-j}+\sum_{j=1}^{k(i)}W_jx_{i-j}\\
&=\sum_{j=0}^{k(i)}W_jx_{i-j}-\sum_{j=k(i+1)}^{k(i)}W_{j+1}x_{i-j}-\sum_{j=0}^{k(i+1)-1}W_{j+1}x_{i-j}\\
&=\sum_{j=0}^{k(i)}(W_j-W_{j+1})x_{i-j}=\sum_{j=0}^{k(i)}A_jx_{i-j}.
\end{aligned}$$

In this case the auxiliary equation (Step 2) has the form $x_{i+1}=W_0x_i$. Let us suppose that for some positive definite matrix Q_0 the matrix equation $W_0'DW_0-D=-Q_0$ has a positive definite solution D. Then the function $v_i=x_i'Dx_i$ is a Lyapunov function for the auxiliary equation.

We will construct the Lyapunov functional V_i in the form $V_i=V_{1i}+V_{2i}$, where (Step 3) $V_{1i}=(x_i-F_3(i))'D(x_i-F_3(i))$. Calculating $\mathbf{E}\Delta V_{1i}$ and using the representations for x_{i+1} and $F_1(i,x_i)$ we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i}&=\mathbf{E}\big[\big(x_{i+1}-F_3(i+1)\big)'D\big(x_{i+1}-F_3(i+1)\big)-V_{1i}\big]\\
&=\mathbf{E}\big(x_{i+1}-F_3(i+1)-x_i+F_3(i)\big)'D\big(x_{i+1}-F_3(i+1)+x_i-F_3(i)\big)\\
&=\mathbf{E}\big((W_0-I)x_i+F_2(i)+G_2(i)\xi_{i+1}\big)'D\big((W_0+I)x_i\\
&\quad+F_2(i)-2F_3(i)+G_2(i)\xi_{i+1}\big)\\
&=\mathbf{E}\left[x_i'(W_0-I)'D(W_0+I)x_i+\sum_{j=1}^{4}I_j\right],
\end{aligned}$$

where

$$I_1=2F_2'(i)D\big(W_0x_i-F_3(i)\big),\qquad I_2=F_2'(i)DF_2(i),$$
$$I_3=2x_i'(I-W_0)'DF_3(i),\qquad I_4=G_{2i}'DG_2(i).$$

Via Lemma 1.3 using some positive definite matrices P_{jl} for I_1 we get

$$\begin{aligned}
I_1&=\sum_{j=k(i+1)}^{k(i)}\sum_{l=0}^{k(i)}\big(x_{i-j}'W_{j+1}'DW_lx_{i-l}+x_{i-l}'W_l'DW_{j+1}x_{i-j}\big)\\
&\le\sum_{j=k(i+1)}^{k(i)}\sum_{l=0}^{k(i)}\big(x_{i-j}'W_{j+1}'P_{jl}W_{j+1}x_{i-j}+x_{i-l}'W_l'DP_{jl}^{-1}DW_lx_{i-l}\big)
\end{aligned}$$

$$= \sum_{j=k(i+1)}^{k(i)} x'_{i-j} W'_{j+1} \sum_{l=0}^{k(i)} P_{jl} W_{j+1} x_{i-j} + \sum_{j=0}^{k(i)} x'_{i-j} W'_j D \sum_{l=k(i+1)}^{k(i)} P_{lj}^{-1} D W_j x_{i-j},$$

using some positive definite matrices R_{jl} for I_2 we get

$$\begin{aligned}
I_2 &= \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} \big(x'_{i-j} W'_{j+1} D W_{l+1} x_{i-l} + x'_{i-l} W'_{l+1} D W_{j+1} x_{i-j}\big) \\
&\le \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} \big(x'_{i-j} W'_{j+1} R_{jl} W_{j+1} x_{i-j} + x'_{i-l} W'_{l+1} D R_{jl}^{-1} D W_{l+1} x_{i-l}\big) \\
&= \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} \big(x'_{i-j} W'_{j+1} R_{jl} W_{j+1} x_{i-j} + x'_{i-j} W'_{j+1} D R_{lj}^{-1} D W_{j+1} x_{i-j}\big) \\
&= \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} x'_{i-j} W'_{j+1} \sum_{l=k(i+1)}^{k(i)} \big(R_{jl} + D R_{lj}^{-1} D\big) W_{j+1} x_{i-j},
\end{aligned}$$

using some positive definite matrices S_j for I_3 we get

$$\begin{aligned}
I_3 &= \sum_{j=1}^{k(i)} \big(x'_i (I - W_0)' D W_j x_{i-j} + x'_{i-j} W'_j D (I - W_0) x_i\big) \\
&\le \sum_{j=1}^{k(i)} \big(x'_{i-j} W'_j S_j W_j x_{i-j} + x'_i (I - W_0)' D S_j^{-1} D (I - W_0) x_i\big),
\end{aligned}$$

and using some positive definite matrices T_{jl} for I_4 we get

$$\begin{aligned}
I_4 &= \frac{1}{2} \sum_{j=0}^{m(i)} \sum_{l=0}^{m(i)} \big(x'_{i-j} B'_j D B_l x_{i-l} + x'_{i-l} B'_l D B_j x_{i-j}\big) \\
&\le \frac{1}{2} \sum_{j=0}^{m(i)} \sum_{l=0}^{m(i)} \big(x'_{i-l} B'_l T_{jl} B_l x_{i-l} + x'_{i-j} B'_j D T_{jl}^{-1} D B_j x_{i-j}\big) \\
&= \frac{1}{2} \sum_{j=0}^{m(i)} x'_{i-j} B'_j \sum_{l=0}^{m(i)} \big(T_{jl} + D T_{lj}^{-1} D\big) B_j x_{i-j}.
\end{aligned}$$

As a result we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} \leq \mathbf{E}\Bigg[x_i'\Bigg((W_0-I)'D(W_0+I)+W_0'D\sum_{l=k_m}^{\hat{k}}P_{l0}^{-1}DW_0\\ +(I-W_0)'D\sum_{l=1}^{\hat{k}}S_l^{-1}D(I-W_0)+\frac{1}{2}\sum_{l=0}^{\hat{m}}B_0'\big(T_{0l}+DT_{l0}^{-1}D\big)B_0\Bigg)x_i\\ +J_{1i}+J_{2i}+J_{3i}\Bigg],\end{aligned} \tag{6.31}$$

where J_{1i}, J_{2i} are defined by (6.23) with

$$\begin{aligned}K_j &= W_j'\Bigg(S_j+D\sum_{l=k_m}^{\hat{k}}P_{lj}^{-1}D\Bigg)W_j,\\ L_j &= \frac{1}{2}\sum_{l=0}^{\hat{m}}B_j'\big(T_{jl}+DT_{lj}^{-1}D\big)B_j,\end{aligned} \tag{6.32}$$

and

$$J_{3i}=\sum_{j=k_m}^{k(i)}x_{i-j}'M_jx_{i-j},\quad M_j=W_{j+1}'\Bigg(\sum_{l=1}^{\hat{k}}P_{jl}+\frac{1}{2}\sum_{l=k_m}^{\hat{k}}\big(R_{jl}+DR_{lj}^{-1}D\big)\Bigg)W_{j+1}.$$

Choose the additional functional V_{2i} (Step 4) in the form $V_{2i}=V_{2i}^{(1)}+V_{2i}^{(2)}+V_{2i}^{(3)}$. Here the functionals $V_{2i}^{(1)}$, $V_{2i}^{(2)}$ are defined by (6.24), (6.25) and (6.32). Put also

$$Z_j^{(3)}=\sum_{l=j}^{\hat{k}}M_l$$

and

$$V_{2i}^{(3)}=\begin{cases}\sum_{j=1}^{k_m-1}x_{i-j}'Z_{k_m}^{(3)}x_{i-j}+\sum_{j=k_m}^{k(i)}x_{i-j}'Z_j^{(3)}x_{i-j}, & k_m>1,\\ \sum_{j=1}^{k(i)}x_{i-j}'Z_j^{(3)}x_{i-j}, & 0\leq k_m\leq 1.\end{cases}$$

The functionals $V_{2i}^{(1)}$ and $V_{2i}^{(2)}$ satisfy the inequalities (6.26) and (6.27). Let us obtain a similar condition for $V_{2i}^{(3)}$.

Firstly suppose that $k_m > 1$. Then using (6.21) we get

$$\begin{aligned}
\Delta V_{2i}^{(3)} &= \sum_{j=1}^{k_m-1} x'_{i+1-j} Z_{k_m}^{(3)} x_{i+1-j} + \sum_{j=k_m}^{k(i+1)} x'_{i+1-j} Z_j^{(3)} x_{i+1-j} \\
&\quad - \sum_{j=1}^{k_m-1} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} - \sum_{j=k_m}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} \\
&= x'_i Z_{k_m}^{(3)} x_i + \sum_{j=k_m}^{k(i+1)-1} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - \sum_{j=k_m}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} \\
&\quad + x'_{i+1-k_m} Z_{k_m}^{(3)} x_{i+1-k_m} + \sum_{j=1}^{k_m-2} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} - \sum_{j=1}^{k_m-1} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} \\
&= x'_i Z_{k_m}^{(3)} x_i + \sum_{j=k_m}^{k(i+1)-1} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - \sum_{j=k_m}^{k(i)} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - J_{3i} \\
&\le x'_i Z_{k_m}^{(3)} x_i - J_{3i}.
\end{aligned}$$

Let now $0 \le k_m \le 1$. Then analogously to (6.26) we have

$$\Delta V_{2i}^{(3)} \le x'_i Z_1^{(3)} x_i - \sum_{j=1}^{k(i)} x'_{i-j} M_j x_{i-j} = x'_i Z_{k_m}^{(3)} x_i - J_{3i}.$$

In this way for $k_m \ge 0$ we have

$$\Delta V_{2i}^{(3)} \le x'_i Z_{k_m}^{(3)} x_i - J_{3i}. \tag{6.33}$$

As a result from (6.31), (6.26), (6.27) and (6.33) for the functional $V_i = V_{1i} + V_{2i}$ we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_i \le \mathbf{E} x'_i \Bigg(&(W_0 - I)' D (W_0 + I) + W'_0 D \sum_{l=k_m}^{\hat{k}} P_{l0}^{-1} D W_0 \\
&+ (I - W_0)' D \sum_{j=1}^{\hat{k}} S_j^{-1} D (I - W_0) \\
&+ \frac{1}{2} \sum_{l=0}^{\hat{m}} B'_0 \big(T_{0l} + D T_{l0}^{-1} D\big) B_0 + Z_1^{(1)} + Z_1^{(2)} + Z_{k_m}^{(3)} \Bigg) x_i
\end{aligned}$$

or (6.7) with

$$
\begin{aligned}
-Q &= (W_0 - I)'D(W_0 + I) + (I - W_0)'D\sum_{j=1}^{\hat{k}} S_j^{-1} D(I - W_0) \\
&\quad + \sum_{j=1}^{\hat{k}} W_j' S_j W_j + \sum_{j=0}^{\hat{k}} W_j' D \sum_{l=k_m}^{\hat{k}} P_{lj}^{-1} D W_j \\
&\quad + \frac{1}{2}\sum_{j=0}^{\hat{m}}\sum_{l=0}^{\hat{m}} B_j' \big(T_{jl} + D T_{lj}^{-1} D\big) B_j \\
&\quad + \sum_{j=k_m}^{\hat{k}} W_{j+1}' \left(\sum_{l=0}^{\hat{k}} P_{jl} + \frac{1}{2}\sum_{l=k_m}^{\hat{k}} \big(R_{jl} + D R_{lj}^{-1} D\big)\right) W_{j+1}. \qquad (6.34)
\end{aligned}
$$

From this and Theorem 1.1 a theorem follows.

Theorem 6.4 *Let for some positive definite matrices Q, P_{jl}, R_{jl}, S_j and T_{jl} the positive definite solution of matrix Riccati equation* (6.34) *exist. Then the trivial solution of* (6.30) *is asymptotically mean square stable.*

Remark 6.6 Analogously with Remarks 6.1 and 6.3 we can show that instead of (6.34) other matrix Riccati equations can be used.

Remark 6.7 Let $k(i) = k = \text{const}$. In this case (6.34) has the form

$$
\begin{aligned}
-Q &= (W_0 - I)'D(W_0 + I) + (I - W_0)'D\sum_{j=1}^{k} S_j^{-1} D(I - W_0) \\
&\quad + \sum_{j=1}^{k} W_j' S_j W_j + \frac{1}{2}\sum_{j=0}^{\hat{m}}\sum_{l=0}^{\hat{m}} B_j' \big(T_{jl} + D T_{lj}^{-1} D\big) B_j.
\end{aligned}
$$

Remark 6.8 It is easy to show that in the scalar case a positive solution of (6.34) exists if and only if

$$
2\alpha_0\alpha_{k_m+1} + \alpha_{k_m+1}^2 + \sigma^2 < (1 - W_0)(1 + W_0 - 2\alpha_1), \quad |W_0| < 1, \qquad (6.35)
$$

where

$$
\alpha_l = \sum_{j=l}^{\hat{k}} |W_j|, \qquad \sigma = \sum_{j=0}^{\hat{m}} |B_j|.
$$

If $k(i) = k = \text{const}$ then $\alpha_{k_m+1} = 0$. For (6.1) in this case we have $W_0 = A + B$, $\alpha_1 = k|B|$, $\sigma = |C|$ and condition (6.35) coincides with (6.19).

Example 6.4 Put in (6.30) $k(i) = [qi]$, where $i \in Z$, $0 < q \leq 1$; $[x]$ is the integral part of a number x. It is easy to see that the function $k(i)$ satisfies condition (6.21), $k_m = 0$, $\hat{k} = \infty$.

Chapter 7
Nonlinear Systems

In this chapter the procedure of the construction of Lyapunov functionals considered above is used for different types of nonlinear systems and for different types of stability.

7.1 Asymptotic Mean Square Stability

Here asymptotic mean square stability conditions are obtained for some stationary and nonstationary equations.

7.1.1 Stationary Systems

Let us show that the procedure of the construction of Lyapunov functionals described above can be used for the investigation of stability in the first approximation. Consider the nonlinear equation

$$x_{i+1} = Ax_i + \sum_{j=-h}^{i} F(i-j, x_j) + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma(i-j, j-l, x_l)\xi_{j+1}, \quad i \in Z, \tag{7.1}$$

where ξ_i, $i \in Z$, is a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, $x_i \in \mathbf{R}^n$, A is a $n \times n$-matrix, the functions $F(i,x)$ and $\sigma(i,j,x)$ satisfy the conditions

$$\big|F(i,x)\big| \le a_i|x|, \qquad \big|\sigma(i,j,x)\big| \le b_{ij}|x|, \tag{7.2}$$

and

$$\alpha_0 = \sum_{i=0}^{\infty} a_i < \infty, \qquad \beta_0 = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} b_{ij}\right)^2 < \infty, \qquad \beta_1 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} b_{ij} < \infty. \tag{7.3}$$

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*, DOI 10.1007/978-0-85729-685-6_7, © Springer-Verlag London Limited 2011

Theorem 7.1 *Let the functions $F(i,x)$ and $\sigma(i,j,x)$ satisfy the conditions* (7.2) *and* (7.3) *and the matrix equation*

$$A'DA - D = -I \tag{7.4}$$

(I *is the identity* $n \times n$*-matrix) has a positive definite solution* D *such that*

$$2(\alpha_0 + \beta_1)\|DA\| + \left[\alpha_0^2 + 2\alpha_0\beta_1 + \beta_0\right]\|D\| < 1, \tag{7.5}$$

where $\|\cdot\|$ *is the operator norm of a matrix. Then the trivial solution of* (7.1) *is asymptotically mean square stable.*

Proof Consider the functional $V_i = x_i' D x_i$, where the matrix D is a positive definite solution of (7.4). Calculating $\mathbf{E}\Delta V_i$ via (7.4) we get

$$\begin{aligned}
\mathbf{E}\Delta V_i = \mathbf{E}\Bigg[&\left(Ax_i + \sum_{j=-h}^{i} F(i-j,x_j) + \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma(i-j,j-l,x_l)\xi_{j+1}\right)' D \\
&\times \left(Ax_i + \sum_{j=-h}^{i} F(i-j,x_j) + \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma(i-j,j-l,x_l)\xi_{j+1}\right) \\
&- x_i' D x_i\Bigg] = -\mathbf{E}|x_i|^2 + \sum_{l=1}^{5} I_l,
\end{aligned} \tag{7.6}$$

where

$$I_1 = \sum_{j=-h}^{i}\sum_{l=-h}^{i} \mathbf{E}F'(i-j,x_j)DF(i-l,x_l),$$

$$I_2 = \sum_{j=0}^{i}\sum_{p=0}^{i}\sum_{l=-h}^{j}\sum_{q=-h}^{p} \mathbf{E}\sigma'(i-j,j-l,x_l)D\sigma(i-p,p-q,x_q)\xi_{p+1}\xi_{j+1},$$

$$I_3 = 2\sum_{j=-h}^{i} \mathbf{E}F'(i-j,x_j)DAx_i,$$

$$I_4 = 2\sum_{j=0}^{i}\sum_{l=-h}^{j} \mathbf{E}\sigma'(i-j,j-l,x_l)DAx_i\xi_{j+1},$$

$$I_5 = 2\sum_{j=0}^{i}\sum_{p=-h}^{i}\sum_{l=-h}^{j} \mathbf{E}F'(i-p,x_p)D\sigma(i-j,j-l,x_l)\xi_{j+1}.$$

Using (7.2) and (7.3) we have

$$
\begin{aligned}
I_1 &\le \|D\|\mathbf{E}\left(\sum_{j=-h}^{i}\left|F(i-j,x_j)\right|\right)^2 \le \|D\|\mathbf{E}\left(\sum_{j=-h}^{i} a_{i-j}|x_j|\right)^2 \\
&\le \|D\| \sum_{j=-h}^{i} a_{i-j} \sum_{j=-h}^{i} a_{i-j}\mathbf{E}|x_j|^2 \le \alpha_0\|D\| \sum_{j=-h}^{i} a_{i-j}\mathbf{E}|x_j|^2 \\
&= \alpha_0\|D\|\left(a_0\mathbf{E}|x_i|^2 + \sum_{j=-h}^{i-1} a_{i-j}\mathbf{E}|x_j|^2\right). \qquad (7.7)
\end{aligned}
$$

Via the properties of ξ_j and (7.2) for I_2 we obtain

$$
\begin{aligned}
I_2 &= \sum_{j=0}^{i}\sum_{l=-h}^{j}\sum_{q=-h}^{j} \mathbf{E}\sigma'(i-j,j-l,x_l)D\sigma(i-j,j-q,x_q) \\
&\le \|D\|\mathbf{E}\sum_{j=0}^{i}\left(\sum_{l=-h}^{j} b_{i-j,j-l}|x_l|\right)^2 \\
&\le \|D\|\sum_{j=0}^{i}\left(\sum_{l=-h}^{j} b_{i-j,j-l}\right)\left(\sum_{l=-h}^{j} b_{i-j,j-l}\mathbf{E}|x_l|^2\right) \\
&= \|D\|\sum_{p=0}^{i}\left(\sum_{l=-h}^{i-p} b_{p,i-p-l}\right)\left(\sum_{l=-h}^{i-p} b_{p,i-p-l}\mathbf{E}|x_l|^2\right) \\
&\le \|D\|\sum_{p=0}^{i}\left(\sum_{k=0}^{\infty} b_{pk}\right)\left(\sum_{l=-h}^{i-p} b_{p,i-p-l}\mathbf{E}|x_l|^2\right) \\
&= \|D\|\sum_{l=-h}^{i}\left(\sum_{p=0}^{i-k_l} b_{p,i-p-l}\left(\sum_{k=0}^{\infty} b_{pk}\right)\right)\mathbf{E}|x_l|^2,
\end{aligned}
$$

where $k_l = \max(0,l)$. Therefore,

$$
I_2 \le \|D\|\left[b_{00}\left(\sum_{k=0}^{\infty} b_{0k}\right)\mathbf{E}|x_i|^2 + \sum_{l=-h}^{i-1}\left(\sum_{p=0}^{i-k_l} b_{p,i-p-l}\left(\sum_{k=0}^{\infty} b_{pk}\right)\right)\mathbf{E}|x_l|^2\right]. \qquad (7.8)
$$

Using (7.2) and (7.3) we get

$$I_3 \le \|DA\| \sum_{j=-h}^{i} a_{i-j}\big(\mathbf{E}|x_i|^2 + \mathbf{E}|x_j|^2\big)$$

$$\le \|DA\|\left((\alpha_0 + a_0)\mathbf{E}|x_i|^2 + \sum_{j=-h}^{i-1} a_{i-j}\mathbf{E}|x_j|^2\right). \tag{7.9}$$

Since $\mathbf{E}\sum_{l=-h}^{i}\sigma'(0, i-l, x_l)DAx_i\xi_{i+1} = 0$, then via (7.2) and (7.3) we have

$$I_4 = 2\sum_{j=0}^{i-1}\sum_{l=-h}^{j} \mathbf{E}\sigma'(i-j, j-l, x_l)DAx_i\xi_{j+1}$$

$$\le \|DA\|\sum_{j=0}^{i-1}\sum_{l=-h}^{j} b_{i-j,j-l}\big(\mathbf{E}|x_i|^2 + \mathbf{E}|x_l|^2\mathbf{E}|\xi_{j+1}|^2\big)$$

$$\le \|DA\|\left(\beta_1\mathbf{E}|x_i|^2 + \sum_{l=-h}^{i-1}\sum_{j=k_l}^{i-1} b_{i-j,j-l}\mathbf{E}|x_l|^2\right). \tag{7.10}$$

Analogously, using (7.2) and (7.3) we get

$$I_5 = 2\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\sum_{p=-h}^{i} \mathbf{E}F'(i-p, x_p)D\sigma(i-j, j-l, x_l)\xi_{j+1}$$

$$\le \|D\|\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\sum_{p=-h}^{i} a_{i-p}b_{i-j,j-l}\big(\mathbf{E}|x_p|^2 + \mathbf{E}|x_l|^2\mathbf{E}|\xi_{j+1}|^2\big)$$

$$= \|D\|\left[\sum_{p=-h}^{i}\left(\sum_{j=0}^{i-1}\sum_{l=-h}^{j} b_{i-j,j-l}\right)a_{i-p}\mathbf{E}|x_p|^2\right.$$

$$\left. + \sum_{l=-h}^{i-1}\sum_{j=k_l}^{i-1} b_{i-j,j-l}\left(\sum_{p=-h}^{i} a_{i-p}\right)\mathbf{E}|x_l|^2\right]$$

$$\le \|D\|\left(\beta_1 a_0\mathbf{E}|x_i|^2 + \beta_1\sum_{p=-h}^{i-1} a_{i-p}\mathbf{E}|x_p|^2 + \alpha_0\sum_{l=-h}^{i-1}\sum_{j=k_l}^{i-1} b_{i-j,j-l}\mathbf{E}|x_l|^2\right). \tag{7.11}$$

As a result of (7.7)–(7.11) it follows that

$$\mathbf{E}\Delta V_i \le a\mathbf{E}|x_i|^2 + \sum_{l=-h}^{i-1} R_{il}\mathbf{E}|x_l|^2,$$

where

$$a = -1 + \left((\alpha_0 + \beta_1) a_0 + b_{00} \sum_{k=0}^{\infty} b_{0k} \right) \|D\| + (\alpha_0 + a_0 + \beta_1) \|DA\|, \qquad R_{ii} = 0,$$

$$R_{il} = \big[(\alpha_0 + \beta_1) \|D\| + \|DA\| \big] a_{i-l} + \|D\| P_{il} + \big(\|DA\| + \alpha_0 \|D\| \big) Q_{il}, \quad l < i,$$

$$P_{il} = \sum_{p=0}^{i-k_l} b_{p,i-p-l} \left(\sum_{k=0}^{\infty} b_{pk} \right), \qquad Q_{il} = \sum_{j=k_l}^{i-1} b_{i-j,j-l}.$$

Via Theorem 1.2 if $a + b < 0$, where

$$b = \sup_{i \in Z} \sum_{m=i+1}^{\infty} R_{mi},$$

then the trivial solution of (7.1) is asymptotically mean square stable.

Note that via (7.3)

$$\sum_{m=i+1}^{\infty} R_{mi} = \big[(\alpha_0 + \beta_1) \|D\| + \|DA\| \big] (\alpha_0 - a_0)$$
$$+ \|D\| \sum_{m=i+1}^{\infty} P_{mi} + \big(\|DA\| + \alpha_0 \|D\| \big) \sum_{m=i+1}^{\infty} Q_{mi}.$$

Besides of that

$$\begin{aligned}
\sum_{m=i+1}^{\infty} P_{mi} &= \sum_{m=1}^{\infty} \sum_{p=0}^{m} b_{p,m-p} \left(\sum_{k=0}^{\infty} b_{pk} \right) \\
&= \sum_{m=0}^{\infty} \sum_{p=0}^{m} b_{p,m-p} \left(\sum_{k=0}^{\infty} b_{pk} \right) - b_{00} \left(\sum_{k=0}^{\infty} b_{0k} \right) \\
&= \sum_{p=0}^{\infty} \sum_{m=p}^{\infty} b_{p,m-p} \left(\sum_{k=0}^{\infty} b_{pk} \right) - b_{00} \left(\sum_{k=0}^{\infty} b_{0k} \right) \\
&= \sum_{p=0}^{\infty} \left(\sum_{l=0}^{\infty} b_{pl} \right) \left(\sum_{k=0}^{\infty} b_{pk} \right) - b_{00} \left(\sum_{k=0}^{\infty} b_{0k} \right) = \beta_0 - b_{00} \left(\sum_{k=0}^{\infty} b_{0k} \right)
\end{aligned}$$

and

$$\sum_{m=i+1}^{\infty} Q_{mi} = \sum_{m=1}^{\infty} \sum_{j=i}^{m+i-1} b_{m+i-j,j-i}$$
$$= \sum_{m=1}^{\infty} \sum_{l=0}^{m-1} b_{m-l,l} = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} b_{kl} = \beta_1.$$

Thus,

$$\sum_{m=i+1}^{\infty} R_{mi} = \big[(\alpha_0 + \beta_1)\|D\| + \|DA\|\big](\alpha_0 - a_0)$$
$$+ \left(\alpha_0\beta_1 + \beta_0 - b_{00} \sum_{k=0}^{\infty} b_{0k}\right)\|D\| + \beta_1\|DA\|$$

and via condition (7.5) $a + b < 0$. The proof is completed. □

Remark 7.1 Put in (7.1) $A = 0$. Then the solution of matrix equation (7.4) is $D = I$ and condition (7.5) takes the form $\alpha_0^2 + 2\alpha_0\beta_1 + \beta_0 < 1$.

Remark 7.2 If (7.1) is a scalar one then the solution of (7.5) is $D = (1 - A^2)^{-1}$ and condition (7.5) takes the form $(\alpha_0 + |A|)^2 + 2(\alpha_0 + |A|)\beta_1 + \beta_0 < 1$.

Consider now the system of two scalar equations

$$x_{i+1} = \sum_{l=-h}^{i} a_{i-l}x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_{j+1},$$
$$y_{i+1} = cy_i \left(1 + \sum_{j=0}^{i} b_j x_j^2\right)^{-1}, \quad i \in Z. \tag{7.12}$$

Here a_j, b_j and c are known constants.

The first equation of this system is (3.1). The sufficient condition for asymptotic mean square stability of this equation is (see (3.3))

$$\alpha_0^2 + 2\alpha_0 S_1 + S_0 < 1, \tag{7.13}$$

where

$$\alpha_0 = \sum_{l=0}^{\infty} |a_l|, \qquad S_0 = \sum_{p=0}^{\infty} \left(\sum_{l=0}^{\infty} |\sigma_l^p|\right)^2, \qquad S_1 = \sum_{p=1}^{\infty} \sum_{l=0}^{\infty} |\sigma_l^p|.$$

Let us suppose also that

$$b_j \geq 0, \qquad |c| < 1. \tag{7.14}$$

Via (7.12) and (7.14)

$$y_{i+1}^2 - y_i^2 = \left[c^2\left(1 + \sum_{j=0}^{i} b_j x_j^2\right)^{-2} - 1\right] y_i^2 \leq \left(c^2 - 1\right) y_i^2.$$

Similar to Sect. 3.1 for the functional $V_i = x_i^2 + y_i^2$ we have

$$\mathbf{E}\Delta V_i \leq -\mathbf{E}x_i^2 + \left(c^2 - 1\right)\mathbf{E}y_i^2 + \sum_{k=-h}^{i} A_{ik}\mathbf{E}x_k^2,$$

where

$$A_{ik} = (\alpha_0 + S_1)|a_{i-k}| + \alpha_0 \sum_{p=1}^{i-k_m} \left|\sigma_{i-k-p}^p\right| + \sum_{p=0}^{i-k_m} \left|\sigma_{i-k-p}^p\right| \sum_{l=0}^{\infty} \left|\sigma_l^p\right|.$$

As is shown in Sect. 3.1, we have

$$\sup_{i\in Z} \sum_{m=i+1}^{\infty} A_{mi} \leq \alpha_0^2 + 2\alpha_0 S_1 + S_0.$$

Analogously to Theorem 1.2 it can be shown that the conditions (7.13) and (7.14) are sufficient conditions for asymptotic mean square stability of the trivial solution of system (7.12).

7.1.2 Nonstationary Systems with Monotone Coefficients

Consider the equation

$$x_{i+1} = -\sum_{j=0}^{i} a_{ij} f(x_j) + \sum_{j=0}^{i} \sigma_{ij} g(x_j)\xi_{i+1}, \quad i \in Z, \tag{7.15}$$

where the functions $f(x)$ and $g(x)$ satisfy the conditions

$$0 < c_1 \leq \frac{f(x)}{x} \leq c_2, \quad x \neq 0, \qquad \left|g(x)\right| \leq c_3|x|. \tag{7.16}$$

Put

$$F_{ij}(a,\sigma) = |a_{ij}| \sum_{k=0}^{i} |\sigma_{ik}|, \quad G(a,\sigma) = \frac{1}{2} \sup_{i\in Z} \sum_{j=i}^{\infty} \left(F_{ji}(a,\sigma) + F_{ji}(\sigma,a)\right). \tag{7.17}$$

Theorem 7.2 *Let the conditions* (7.16) *hold and the coefficients* a_{ij}, σ_{ij}, $i \in Z$, $j = 0, \dots, i$, *satisfy the conditions*

$$a_{ij} \geq a_{i,j-1} \geq 0, \tag{7.18}$$

$$a_{i+1,j-1} - a_{i+1,j} - a_{i,j-1} + a_{ij} \geq 0, \tag{7.19}$$

$$\begin{aligned} a &= \sup_{i \in Z}(a_{i+1,i+1} + a_{ii} - a_{i+1,i}) \\ &< 2\left[\frac{1}{c_2} - \frac{c_3}{c_1}\left(c_3 G(\sigma, \sigma) + \frac{c_2 - c_1}{2} G(a, \sigma)\right)\right], \end{aligned} \tag{7.20}$$

$$G(\sigma, \sigma) < \infty, \; G(a, \sigma) < \infty \tag{7.21}$$

(*here it is supposed that* $a_{i,-1} = 0$). *Then the trivial solution of* (7.15) *is asymptotically mean square stable.*

Proof Following the procedure of the construction of the Lyapunov functionals consider the auxiliary equation in the form $y_{i+1} = 0$, $i \in Z$. The function $v_i = y_i f(y_i)$ is a Lyapunov function for this system, since via (7.16)

$$\Delta v_i = y_{i+1} f(y_{i+1}) - y_i f(y_i) = -y_i f(y_i) \leq -c_1 y_i^2.$$

We will construct a Lyapunov functional for (7.15) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{1i} = x_i f(x_i), \quad V_{2i} = \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} f(x_k)\right)^2,$$

and the numbers α_{ij} are defined by

$$\alpha_{ij} = \frac{a_{ij} - a_{i,j-1}}{2 - a c_2}, \quad i \in Z, \; j = 0, 1, \dots, i+1. \tag{7.22}$$

Here it is supposed that $a_{i,-1} = 0$, $a_{i,i+1} = a$.

From (7.18)–(7.20) and (7.22) it follows that the numbers α_{ij} satisfy the conditions

$$0 \leq \alpha_{i+1,j} \leq \alpha_{ij}, \quad i \in Z, \; j = 0, 1, \dots, i+1. \tag{7.23}$$

Via (7.16) we have

$$\Delta V_{1i} = -x_i f(x_i) + x_{i+1} f(x_{i+1}) \leq -c_1 x_i^2 + x_{i+1} f(x_{i+1}).$$

Representing ΔV_{2i} in the form

$$\begin{aligned}\Delta V_{2i} &= \sum_{j=0}^{i+1}\alpha_{i+1,j}\left(\sum_{k=j}^{i+1} f(x_k)\right)^2 - \sum_{j=0}^{i}\alpha_{ij}\left(\sum_{k=j}^{i} f(x_k)\right)^2\\ &= \sum_{j=0}^{i+1}(\alpha_{i+1,j}-\alpha_{ij})\left(\sum_{k=j}^{i+1} f(x_k)\right)^2\\ &\quad + \sum_{j=0}^{i}\alpha_{ij}\left[\left(\sum_{k=j}^{i+1} f(x_k)\right)^2 - \left(\sum_{k=j}^{i} f(x_k)\right)^2\right] + \alpha_{i,i+1}f^2(x_{i+1})\end{aligned}$$

and using (7.23), we get

$$\begin{aligned}\Delta V_{2i} &\leq \sum_{j=0}^{i}\alpha_{ij}\left[2f(x_{i+1})\sum_{k=j}^{i} f(x_k) + f^2(x_{i+1})\right] + \alpha_{i,i+1}f^2(x_{i+1})\\ &= f^2(x_{i+1})\sum_{j=0}^{i+1}\alpha_{ij} + 2f(x_{i+1})\sum_{k=0}^{i} f(x_k)\sum_{j=0}^{k}\alpha_{ij}.\end{aligned}$$

From (7.22) it follows that

$$\sum_{j=0}^{k}\alpha_{ij} = \sum_{j=0}^{k}\frac{a_{ij}-a_{i,j-1}}{2-ac_2} = \frac{a_{ik}}{2-ac_2}.$$

Therefore using (7.16) and (7.15) by $x_{i+1} \neq 0$ we obtain

$$\begin{aligned}\Delta V_{2i} &\leq \frac{af^2(x_{i+1})}{2-ac_2} + \frac{2f(x_{i+1})}{2-ac_2}\sum_{k=0}^{i} a_{ik}f(x_k)\\ &= \frac{a}{2-ac_2}\frac{f(x_{i+1})}{x_{i+1}}x_{i+1}f(x_{i+1}) + \frac{2f(x_{i+1})}{2-ac_2}\left(\sum_{j=0}^{i}\sigma_{ij}g(x_j)\xi_{i+1} - x_{i+1}\right)\\ &\leq \frac{ac_2}{2-ac_2}x_{i+1}f(x_{i+1}) + \frac{2f(x_{i+1})}{2-ac_2}\left(\sum_{j=0}^{i}\sigma_{ij}g(x_j)\xi_{i+1} - x_{i+1}\right)\\ &= -x_{i+1}f(x_{i+1}) + \frac{2}{2-ac_2}\sum_{j=0}^{i}\sigma_{ij}g(x_j)\xi_{i+1}f(x_{i+1}).\end{aligned}$$

As a result for the functional V_i we have

$$\mathbf{E}\Delta V_i \leq -c_1\mathbf{E}x_i^2 + \frac{2}{2-ac_2}\zeta_i, \tag{7.24}$$

where

$$\zeta_i = \sum_{j=0}^{i} \sigma_{ij} \mathbf{E} g(x_j) \xi_{i+1} f(x_{i+1}). \tag{7.25}$$

Let us estimate $|\zeta_i|$. Using (7.15) by $j \le i$ we obtain

$$\begin{aligned}
&\mathbf{E} g(x_j)\xi_{i+1} f(x_{i+1}) \\
&\quad = \mathbf{E} g(x_j)\mathbf{E}\left(x_{i+1} \frac{f(x_{i+1})}{x_{i+1}} \xi_{i+1}/\mathfrak{F}_i \right) \\
&\quad = \mathbf{E} g(x_j)\left[-\sum_{k=0}^{i} a_{ik} f(x_k) \mathbf{E}\left(\frac{f(x_{i+1})}{x_{i+1}} \xi_{i+1}/\mathfrak{F}_i \right) \right. \\
&\qquad \left. + \sum_{k=0}^{i} \sigma_{ik} g(x_k) \mathbf{E}\left(\frac{f(x_{i+1})}{x_{i+1}} \xi_{i+1}^2/\mathfrak{F}_i \right) \right].
\end{aligned}$$

Via (7.16)

$$\mathbf{E}\left(\frac{f(x_{i+1})}{x_{i+1}} \xi_{i+1}^2/\mathfrak{F}_i \right) \le c_2.$$

From this and (7.25) it follows that

$$|\zeta_i| \le \mathbf{E}\left| \sum_{j=0}^{i} \sigma_{ij} g(x_j) \right| \left| \sum_{k=0}^{i} a_{ik} f(x_k) \right| |\mu_i| + c_2 \mathbf{E}\left(\sum_{j=0}^{i} \sigma_{ij} g(x_j) \right)^2, \tag{7.26}$$

where

$$\mu_i = \mathbf{E}(\eta_{i+1}\xi_{i+1}/\mathfrak{F}_i), \quad \eta_i = \frac{f(x_i)}{x_i}. \tag{7.27}$$

To estimate μ_i consider the following.

Lemma 7.1 *Let ξ_i and η_i, $i \in Z$, be sequences of $\mathfrak{F}_i$-adapted random variables such that $\mathbf{E}(\xi_{i+1}/\mathfrak{F}_i) = 0$, $\mathbf{E}(\xi_{i+1}^2/\mathfrak{F}_i) = 1$, $0 < c_1 \le \eta_i \le c_2$. Then the sequence $\mu_i = \mathbf{E}(\eta_{i+1}\xi_{i+1}/\mathfrak{F}_i)$ satisfies the condition*

$$|\mu_i| \le \frac{c_2 - c_1}{2}. \tag{7.28}$$

Proof Let $\mathbf{P}_i$ be a measure corresponding to the conditional expectation $\mathbf{E}(./\mathfrak{F}_i)$. Let $\Omega^+ = \{\omega : \xi_i(\omega) \ge 0\}$, $\Omega^- = \{\omega : \xi_i(\omega) < 0\}$. Then

$$\begin{aligned}
\mu_i &= \int_{\Omega} \eta_{i+1}(\omega)\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega) \\
&= \int_{\Omega^+} \eta_{i+1}(\omega)\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega) + \int_{\Omega^-} \eta_{i+1}(\omega)\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega) \\
&\le c_2 \int_{\Omega^+} \xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega) + c_1 \int_{\Omega^-} \xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega).
\end{aligned}$$

Since

$$\mathbf{E}(\xi_{i+1}/\mathfrak{F}_i)=\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)+\int_{\Omega^-}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)=0,$$

we have

$$\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)=-\int_{\Omega^-}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)=\int_{\Omega^-}\left|\xi_{i+1}(\omega)\right|\mathbf{P}_i(\mathrm{d}\omega)$$

and

$$\begin{aligned}\mathbf{E}\bigl(|\xi_{i+1}|/\mathfrak{F}_i\bigr)&=\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)+\int_{\Omega^-}\left|\xi_{i+1}(\omega)\right|\mathbf{P}_i(\mathrm{d}\omega)\\&=2\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega).\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_i&\le(c_2-c_1)\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)=\frac{c_2-c_1}{2}\mathbf{E}\bigl(|\xi_{i+1}|/\mathfrak{F}_i\bigr)\\&\le\frac{c_2-c_1}{2}\sqrt{\mathbf{E}\bigl(\xi_{i+1}^2/\mathfrak{F}_i\bigr)}=\frac{c_2-c_1}{2}.\end{aligned}$$

Similarly

$$\begin{aligned}\mu_i&=\int_{\Omega^+}\eta_{i+1}(\omega)\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)+\int_{\Omega^-}\eta_{i+1}(\omega)\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)\\&\ge c_1\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)+c_2\int_{\Omega^-}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)\\&=(c_1-c_2)\int_{\Omega^+}\xi_{i+1}(\omega)\mathbf{P}_i(\mathrm{d}\omega)=\frac{c_1-c_2}{2}\mathbf{E}\bigl(|\xi_{i+1}|/\mathfrak{F}_i\bigr)\\&\ge\frac{c_1-c_2}{2}\sqrt{\mathbf{E}\bigl(\xi_{i+1}^2/\mathfrak{F}_i\bigr)}=\frac{c_1-c_2}{2}.\end{aligned}$$

The proof is completed. □

Via Lemma 7.1 and (7.24) and (7.26)–(7.28)

$$\begin{aligned}\mathbf{E}\Delta V_i\le-c_1\mathbf{E}x_i^2+\frac{2}{2-ac_2}\Biggl[&c_2\mathbf{E}\Biggl(\sum_{j=0}^{i}\sigma_{ij}g(x_j)\Biggr)^2\\&+\frac{c_2-c_1}{2}\mathbf{E}\Biggl|\sum_{j=0}^{i}\sigma_{ij}g(x_j)\Biggr|\Biggl|\sum_{k=0}^{i}a_{ik}f(x_k)\Biggr|\Biggr].\end{aligned}\tag{7.29}$$

Using (7.16) and (7.17) we have

$$\mathbf{E}\left(\sum_{j=0}^{i}\sigma_{ij}g(x_j)\right)^2 \le c_3^2\mathbf{E}\left(\sum_{j=0}^{i}|\sigma_{ij}||x_j|\right)^2 \le c_3^2\sum_{j=0}^{i}F_{ij}(\sigma,\sigma)\mathbf{E}x_j^2, \qquad (7.30)$$

$$\begin{aligned}
&\mathbf{E}\left|\sum_{j=0}^{i}\sigma_{ij}g(x_j)\right|\left|\sum_{k=0}^{i}a_{ik}f(x_k)\right| \\
&\quad\le c_2c_3\sum_{j=0}^{i}\sum_{k=0}^{i}|\sigma_{ij}||a_{ik}|\mathbf{E}|x_j||x_k| \\
&\quad\le \frac{1}{2}c_2c_3\left(\sum_{k=0}^{i}F_{ik}(a,\sigma)\mathbf{E}x_k^2+\sum_{j=0}^{i}F_{ij}(\sigma,a)\mathbf{E}x_j^2\right) \\
&\quad= \frac{1}{2}c_2c_3\sum_{j=0}^{i}\big(F_{ij}(a,\sigma)+F_{ij}(\sigma,a)\big)\mathbf{E}x_j^2. \qquad (7.31)
\end{aligned}$$

Substituting (7.30) and (7.31) into (7.29), we obtain

$$\mathbf{E}\Delta V_i \le -c_1\mathbf{E}x_i^2+\sum_{j=0}^{i}Q_{ij}\mathbf{E}x_j^2,$$

where

$$Q_{ij}=\frac{2c_2c_3}{2-ac_2}\left(c_3F_{ij}(\sigma,\sigma)+\frac{1}{4}(c_2-c_1)\big(F_{ij}(a,\sigma)+F_{ij}(\sigma,a)\big)\right)$$

and via (7.17) and (7.21)

$$\sum_{j=i}^{\infty}Q_{ji}\le\frac{2c_2c_3}{2-ac_2}\left(c_3G(\sigma,\sigma)+\frac{c_2-c_1}{2}G(a,\sigma)\right)<\infty.$$

Via Theorem 1.2 if the inequality

$$\frac{2c_2c_3}{2-ac_2}\left(c_3G(\sigma,\sigma)+\frac{c_2-c_1}{2}G(a,\sigma)\right)<c_1 \qquad (7.32)$$

(which is equivalent to (7.20)) holds then the trivial solution of (7.15) is asymptotically mean square stable. Theorem is proven. □

Remark 7.3 Consider the stationary equation

$$x_{i+1}=-\sum_{j=0}^{i}a_{i-j}f(x_j)+\sum_{j=0}^{i}\sigma_{i-j}g(x_j)\xi_{i+1},\quad i\in Z.$$

It is easy to see that in this case $G(a,\sigma)=a\sigma$, where $a=\sum_{i=0}^{\infty}a_i$, $\sigma=\sum_{i=0}^{\infty}|\sigma_i|$. So the stability conditions (7.18)–(7.20) have the form

$$a_i \ge a_{i+1} \ge 0, \quad a_{i+2}-2a_{i+1}+a_i \ge 0, \quad i=0,1,\dots, \tag{7.33}$$

$$a_0-\frac{a_1}{2}<\frac{1}{c_2}-\frac{c_3}{c_1}\sigma\left(c_3\sigma+\frac{c_2-c_1}{2}a\right). \tag{7.34}$$

Remark 7.4 If $f(x)=g(x)=x$ then $c_1=c_2=c_3=1$ and stability conditions (7.18)–(7.20) and (7.33) and (7.34) coincide with similar conditions (4.31) for the linear case.

Remark 7.5 If the function $f(x)$ is a linear one (i.e. $c_1=c_2$) or in (7.15) stochastic perturbations are absent ($c_3=0$) then the boundedness of $G(a,\sigma)$ (or a in the stationary case) is not supposed.

Remark 7.6 Note that without loss of generality in condition (7.16) we can put $c_3=1$ and $c_1\le c_2=1$ or $c_2\ge c_1=1$. In fact, if it is not so we can put for instance $a_{ij}f(x_j)=\tilde{a}_{ij}\tilde{f}(x_j)$, where $\tilde{a}_{ij}=c_2a_{ij}$, $\tilde{f}(x)=c_2^{-1}f(x)$. In this case the function $\tilde{f}(x)$ satisfies condition (7.16) with $c_2=1$.

Example 7.1 Consider the equation with stationary coefficients

$$x_{i+1}=-\lambda_1\sum_{j=0}^{i}q_1^{i-j}f(x_j)+\lambda_2\sum_{j=0}^{i}q_2^{i-j}g(x_j)\xi_{i+1}, \tag{7.35}$$

where $\lambda_l>0$, $0<q_l<1$, $l=1,2$, functions $f(x)$ and $g(x)$ satisfy conditions (7.16). Equation (7.35) is a particular case of (7.1) with $A=0$ and condition (7.2), where $a_i=\lambda_1c_2q_1^i$, $b_{ij}=0$, $i>0$, $b_{0j}=\lambda_2c_3q_2^j$. Thus, using (7.3) we have

$$\alpha_0=\frac{\lambda_1c_2}{1-q_1}, \qquad \beta_0=\frac{\lambda_2^2c_3^2}{(1-q_2)^2}, \qquad \beta_1=0.$$

Via Remark 7.2 sufficient condition for asymptotic mean square stability of the trivial solution of (7.35) has the form $\alpha_0^2+\beta_0<1$ or

$$\frac{\lambda_1^2c_2^2}{(1-q_1)^2}+\frac{\lambda_2^2c_3^2}{(1-q_2)^2}<1. \tag{7.36}$$

From (7.33) and (7.34) another stability condition follows:

$$\lambda_1-\frac{\lambda_1q_1}{2}<\frac{1}{c_2}-\frac{\lambda_2c_3}{c_1(1-q_2)}\left(\frac{\lambda_2c_3}{1-q_2}+\frac{\lambda_1(c_2-c_1)}{2(1-q_1)}\right)$$

or

$$\frac{\lambda_1}{2}\left[c_1(2-q_1)+\frac{\lambda_2c_3(c_2-c_1)}{(1-q_1)(1-q_2)}\right]+\frac{\lambda_2^2c_3^2}{(1-q_2)^2}<\frac{c_1}{c_2}. \tag{7.37}$$

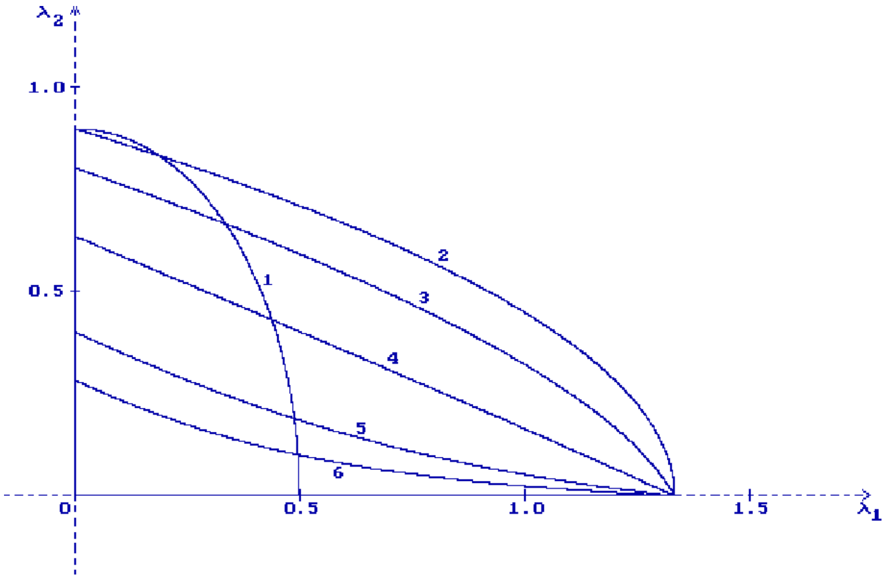


Fig. 7.1 Stability regions for (7.35) obtained by the conditions (7.36) and (7.37): (*1*) $c_2 = c_3 = 1$, $q_1 = 0.5$, $q_2 = 0.1$, (*2*) $c_1 = 1$, (*3*) $c_1 = 0.8$, (*4*) $c_1 = 0.5$, (*5*) $c_1 = 0.2$, (*6*) $c_1 = 0.1$

In Fig. 7.1 stability regions in the space of the parameters (λ_1, λ_2) are shown which are constructed via condition (7.36): (1) $c_2 = c_3 = 1$, $q_1 = 0.5$, $q_2 = 0.1$ and condition (7.37) for the same values of the parameters c_2, c_3, q_1, q_2 and different values of the parameter c_1: (2) $c_1 = 1$, (3) $c_1 = 0.8$, (4) $c_1 = 0.5$, (5) $c_1 = 0.2$, (6) $c_1 = 0.1$.

Example 7.2 Consider the equation with stationary coefficients

$$x_{i+1} = -\lambda_1 \sum_{j=0}^{i} a_{i-j} f(x_j) + \lambda_2 \sum_{j=0}^{i} a_{i-j} g(x_j) \xi_{i+1}, \quad a_i = \frac{1}{(i+1)^2}, \tag{7.38}$$

where $\lambda_l > 0$, $l = 1, 2$, functions $f(x)$ and $g(x)$ satisfy conditions (7.16). Similar to Example 7.1 from Theorems 7.1 and 7.2, respectively, we obtain the two following sufficient conditions for asymptotic mean square stability of the trivial solution of (7.38):

$$\left(\lambda_1^2 c_2^2 + \lambda_2^2 c_3^2\right) S^2 < 1 \tag{7.39}$$

and

$$\frac{\lambda_1}{2}\left(\frac{7}{4}c_1 + \lambda_2 c_3 (c_2 - c_1) S^2\right) + \lambda_2^2 c_3^2 S^2 < \frac{c_1}{c_2}, \tag{7.40}$$

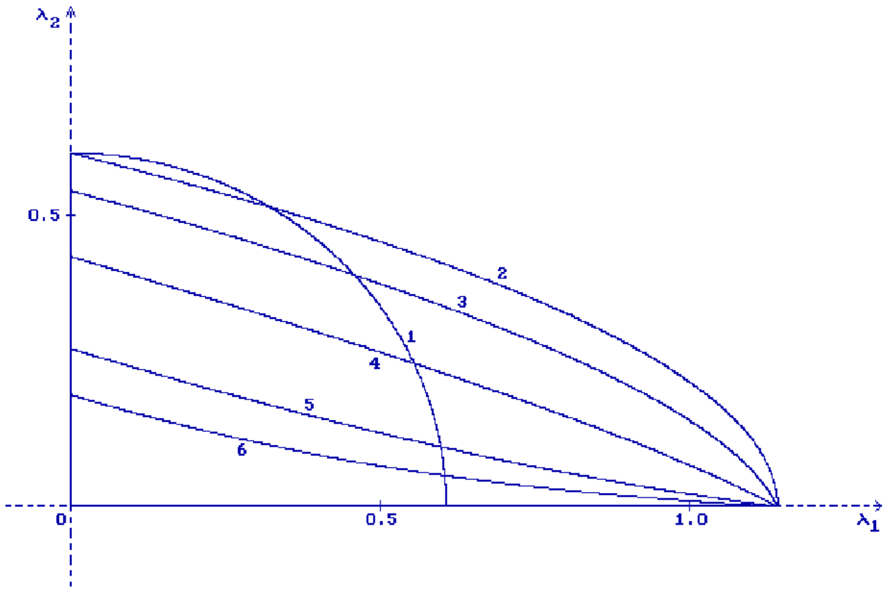


Fig. 7.2 Stability regions for (7.38) obtained by the conditions (7.39) and (7.40): (*1*) $c_2 = c_3 = 1$, (*2*) $c_1 = 1$, (*3*) $c_1 = 0.8$, (*4*) $c_1 = 0.5$, (*5*) $c_1 = 0.2$, (*6*) $c_1 = 0.1$

where

$$S = \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = 1.645.$$

In Fig. 7.2 stability regions in the space of the parameters (λ_1, λ_2) are shown which are constructed via condition (7.39): (1) $c_2 = c_3 = 1$ and condition (7.40) for the same values of the parameters c_2, c_3 and different values of the parameter c_1: (2) $c_1 = 1$, (3) $c_1 = 0.8$, (4) $c_1 = 0.5$, (5) $c_1 = 0.2$, (6) $c_1 = 0.1$.

One can see that for both equations (7.35) and (7.38) the conditions of Theorem 7.2 extend the stability region given by Theorem 7.1.

Example 7.3 Consider now the equation with nonstationary coefficients

$$x_{i+1} = -\lambda_1 \sum_{j=0}^{i} a_{ij} f(x_j) + \lambda_2 \sum_{j=0}^{i} a_{ij} g(x_j) \xi_{i+1}, \quad a_{ij} = \frac{(j+1)}{(i+2)^2}, \tag{7.41}$$

where $\lambda_l > 0$, $l = 1, 2$, functions $f(x)$ and $g(x)$ satisfy conditions (7.16).

Let us construct stability condition for this equation using Theorem 7.2 only, since Theorem 7.1 cannot be used for the nonstationary case.

It is easy to obtain

$$\begin{aligned} G(a,\sigma) &= \lambda_1\lambda_2 \sup_{i\in Z}\left[\sum_{j=i}^{\infty}\frac{i+1}{(j+2)^2}\sum_{k=0}^{j}\frac{k+1}{(j+2)^2}\right] \\ &= \lambda_1\lambda_2 \sup_{i\in Z}\left[(i+1)\sum_{j=i}^{\infty}\frac{1}{(j+2)^4}\frac{(j+2)(j+1)}{2}\right] \\ &\le \frac{\lambda_1\lambda_2}{2}\sup_{i\in Z}\left[(i+1)\sum_{j=i+2}^{\infty}\frac{1}{j^2}\right]. \end{aligned}$$

Via Lemma 1.4

$$\begin{aligned} G(a,\sigma) &\le \frac{\lambda_1\lambda_2}{2}\sup_{i\in Z}\left[(i+1)\left(\frac{1}{(i+2)^2}+\int_{i+2}^{\infty}\frac{\mathrm{d}x}{x^2}\right)\right] \\ &= \frac{\lambda_1\lambda_2}{2}\sup_{i\in Z}\left[(i+1)\left(\frac{1}{(i+2)^2}+\frac{1}{i+2}\right)\right] \\ &= \frac{\lambda_1\lambda_2}{2}\sup_{i\in Z}\left[\frac{(i+1)(i+3)}{(i+2)^2}\right] = \frac{\lambda_1\lambda_2}{2}. \end{aligned}$$

Analogously $G(\sigma,\sigma) \le \frac{1}{2}\lambda_2^2$. Conditions (7.18) and (7.19) hold. Note that

$$\begin{aligned} & a_{i+1,i+1} + a_{ii} - a_{i+1,i} \\ &\quad = \lambda_1\left(\frac{i+2}{(i+3)^2}+\frac{i+1}{(i+2)^2}-\frac{i+1}{(i+3)^2}\right) \\ &\quad = \lambda_1\left(\frac{1}{(i+3)^2}+\frac{i+1}{(i+2)^2}\right). \end{aligned}$$

Since the expression in brackets decrease for $i \in Z$ then supremum is reached by $i = 0$, i.e. $a = \frac{13}{36}\lambda_1$. So the condition (7.20) takes the form

$$\frac{13\lambda_1}{36} < 2\left[\frac{1}{c_2} - \frac{c_3}{c_1}\left(\frac{\lambda_2^2 c_3}{2} + \frac{\lambda_1\lambda_2(c_2-c_1)}{4}\right)\right]$$

or

$$\frac{\lambda_1}{4}\left(\frac{13}{18}c_1 + \lambda_2 c_3(c_2-c_1)\right) + \frac{\lambda_2^2 c_3^2}{2} < \frac{c_1}{c_2}. \tag{7.42}$$

In Fig. 7.3 stability regions in the space of the parameters (λ_1, λ_2) are shown which are constructed via condition (7.42) for $c_2 = c_3 = 1$ and different values of the parameter c_1: (1) $c_1 = 1$, (2) $c_1 = 0.8$, (3) $c_1 = 0.5$, (4) $c_1 = 0.2$, 5) $c_1 = 0.1$.

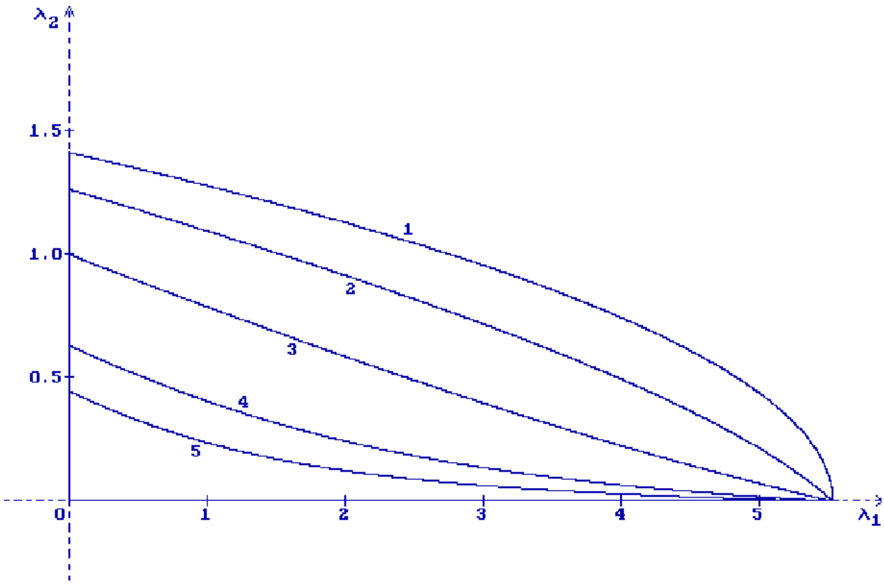


Fig. 7.3 Stability regions for (7.41) obtained by condition (7.42) for $c_2 = c_3 = 1$ and: (*1*) $c_1 = 1$, (*2*) $c_1 = 0.8$, (*3*) $c_1 = 0.5$, (*4*) $c_1 = 0.2$, (*5*) $c_1 = 0.1$

7.2 Stability in Probability

Here it is shown that using the investigation procedure of the construction of the Lyapunov functionals of stability in probability of nonlinear stochastic difference equation with an order of nonlinearity higher than one can be reduced to the investigation of asymptotic mean square stability of the linear part of this equation.

7.2.1 Basic Theorem

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, i be a discrete time, $i \in Z_0 \cup Z$, $Z_0 = \{-h, \dots, 0\}$, $Z = \{0, 1, \dots\}$, h be a given nonnegative number, $\mathfrak{F}_i \in \sigma$, $i \in Z$, be a sequence of σ-algebras, $\mathbf{E}$ be the mathematical expectation, $\mathbf{P}\{./\mathfrak{F}_i\}$ and $\mathbf{E}_i = \mathbf{E}\{./\mathfrak{F}_i\}$ be, respectively, conditional probability and conditional expectation with respect to the σ-algebra $\mathfrak{F}_i$, ξ_i, $i \in Z$, be a sequence of mutually independent $\mathfrak{F}_i$-adapted random variables such that $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, $\|\varphi\|_0 = \max_{j \in Z_0} |\varphi_j|$, $U_\varepsilon = \{x : |x| \le \varepsilon\}$.

Consider the difference equation

$$x_{i+1} = F(i, x_{-h}, \dots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \dots, x_j)\xi_{j+1}, \quad i \in Z, \tag{7.43}$$

with the initial condition

$$x_j = \varphi_j, \quad j \in Z_0. \tag{7.44}$$

Here $F : Z * S \Rightarrow \mathbf{R}^n$, $G : Z * Z * S \Rightarrow \mathbf{R}^n$, S is a space of sequences with elements from $\mathbf{R}^n$. It is assumed that $F(i, \ldots)$ does not depend on x_j for $j > i$, $G(i, j, \ldots)$ does not depend on x_k for $k > j$ and $F(i, 0, \ldots, 0) = 0$, $G(i, j, 0, \ldots, 0) = 0$.

Definition 7.1 The trivial solution of (7.43) with initial condition (7.44) is called stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that the solution $x_i = x_i(\varphi)$ of (7.43) satisfies the inequality

$$\mathbf{P}\Big\{\sup_{i\in Z}|x_i| > \varepsilon/\mathfrak{F}_0\Big\} < \varepsilon_1 \tag{7.45}$$

for any initial function φ which is less than δ with probability 1, i.e.,

$$\mathbf{P}\big\{\|\varphi\|_0 < \delta\big\} = 1. \tag{7.46}$$

Theorem 7.3 *Let there exist a functional $V_i = V(i, x_{-h}, \ldots, x_i)$ which satisfies the conditions*

$$V(i, x_{-h}, \ldots, x_i) \geq c_0|x_i|^2, \tag{7.47}$$

$$V(0, \varphi_{-h}, \ldots, \varphi_0) \leq c_1\|\varphi\|_0^2, \tag{7.48}$$

$$\mathbf{E}_i\Delta V_i \leq 0, \qquad x_j \in U_\varepsilon, \quad -h \leq j \leq i,\ i \in Z, \tag{7.49}$$

where $\varepsilon > 0$, $c_0 > 0$, $c_1 > 0$. *Then the trivial solution of* (7.43) *with the initial condition* (7.44) *is stable in probability.*

Proof We will show that for any positive numbers ε and ε_1 there exists a positive number δ such that the solution of (7.43) satisfies condition (7.45) if initial function (7.44) satisfies condition (7.46).

Let x_i be a solution of (7.43). Consider the random variable τ such that

$$\tau = \inf\big\{i \in Z : |x_i| \leq \varepsilon, |x_{i+1}| > \varepsilon\big\} \tag{7.50}$$

and two events: $\{\sup_{i\in Z}|x_i| > \varepsilon\}$ and $\{|x_{\tau+1}| > \varepsilon\}$. It is clear that

$$\Big\{\sup_{i\in Z}|x_i| > \varepsilon\Big\} \subset \big\{|x_{\tau+1}| > \varepsilon\big\}. \tag{7.51}$$

From (7.49) we have

$$\mathbf{E}_0\sum_{i=0}^{\tau}\mathbf{E}_i\Delta V_i = \sum_{i=0}^{\tau}(\mathbf{E}_0V_{i+1} - \mathbf{E}_0V_i) = \mathbf{E}_0V_{\tau+1} - V_0 \leq 0. \tag{7.52}$$

Using (7.51), the Chebyshev inequality, (7.47), (7.52), (7.48) and (7.46) we get

$$\mathbf{P}\Big\{\sup_{i\in Z}|x_i| > \varepsilon/\mathfrak{F}_0\Big\} \le \mathbf{P}\big\{|x_{\tau+1}| > \varepsilon/\mathfrak{F}_0\big\}$$
$$\le \frac{\mathbf{E}_0|x_{\tau+1}|^2}{\varepsilon^2} \le \frac{\mathbf{E}_0 V_{\tau+1}}{c_0\varepsilon^2} \le \frac{V_0}{c_0\varepsilon^2} \le \frac{c_1\|\varphi\|_0^2}{c_0\varepsilon^2} < \frac{c_1\delta^2}{c_0\varepsilon^2}.$$

Choosing $\delta = \varepsilon\sqrt{\varepsilon_1 c_0 c_1^{-1}}$ we obtain (7.45). The theorem is proven. □

Remark 7.7 It is easy to see that if $\varepsilon \ge \varepsilon_0$ then

$$\mathbf{P}\Big\{\sup_{i\in Z}|x_i| > \varepsilon/f_0\Big\} \le \mathbf{P}\Big\{\sup_{i\in Z}|x_i| > \varepsilon_0/f_0\Big\}.$$

It means that if condition (7.45) holds for small enough $\varepsilon_0 > 0$, then it holds for any $\varepsilon \ge \varepsilon_0$. Thus, for stability in probability of the trivial solution of (7.43) it is sufficient to prove condition (7.45) for small enough $\varepsilon > 0$.

7.2.2 *Quasilinear System with Order of Nonlinearity Higher than One*

Consider the nonlinear scalar difference equation

$$x_{i+1} = \sum_{j=0}^{i+h} a_j x_{i-j} + \sum_{j=0}^{i+h} \sigma_j x_{i-j}\xi_{i+1} + g_i(x_{-h}, \dots, x_i) \tag{7.53}$$

with the initial condition (7.44). It is supposed that

$$\big|g_i(x_{-h}, \dots, x_i)\big| \le \sum_{j=0}^{i+h} \gamma_j |x_{i-j}|^{\nu_j}, \quad \nu_j > 1,\ j \in Z. \tag{7.54}$$

Put

$$a = \sum_{j=0}^{\infty} |a_j|, \qquad \sigma = \sum_{j=0}^{\infty} |\sigma_j|, \qquad \gamma = \sum_{j=0}^{\infty} \gamma_j. \tag{7.55}$$

Theorem 7.4 *Let* $\gamma < \infty$ *and*

$$a^2 + \sigma^2 < 1. \tag{7.56}$$

Then the trivial solution of (7.53) *is stable in probability.*

Proof It is sufficient to construct a Lyapunov functional V_i satisfying the Theorem 7.3 conditions. We will construct this functional in the form $V_i = V_{i1} + V_{i2}$ with $V_{i1} = x_i^2$. Then

$$\begin{aligned}
\mathbf{E}_i \Delta V_{1i} &= \mathbf{E}_i \big(x_{i+1}^2 - x_i^2\big) \\
&= \mathbf{E}_i \left(\sum_{j=0}^{i+h} a_j x_{i-j} + \sum_{j=0}^{i+h} \sigma_j x_{i-j} \xi_{i+1} + g_i(x_{-h}, \dots, x_i) \right)^2 - x_i^2 \\
&= \left(\sum_{j=0}^{i+h} a_j x_{i-j} \right)^2 + \left(\sum_{j=0}^{i+h} \sigma_j x_{i-j} \right)^2 + g_i^2(x_{-h}, \dots, x_i) \\
&\quad + 2 \sum_{j=0}^{i+h} a_j x_{i-j} g_i(x_{-h}, \dots, x_i) - x_i^2.
\end{aligned}$$

Put

$$\mu_k(\varepsilon) = \sum_{j=0}^{\infty} \gamma_j \varepsilon^{k(\nu_j - 1)}, \quad k = 1, 2. \tag{7.57}$$

From (7.55) and (7.57) it follows that if $\varepsilon \le 1$ then $\mu_k(\varepsilon) \le \gamma < \infty$. Using (7.54) and (7.55) and assuming that $x_j \in U_\varepsilon$, $j \le i$, we obtain

$$\left(\sum_{j=0}^{i+h} a_j x_{i-j} \right)^2 \le a \sum_{j=0}^{i+h} |a_j| x_{i-j}^2, \qquad \left(\sum_{j=0}^{i+h} \sigma_j x_{i-j} \right)^2 \le \sigma \sum_{j=0}^{i+h} |\sigma_j| x_{i-j}^2, \tag{7.58}$$

$$g_i^2(x_{-h}, \dots, x_i) \le \gamma \sum_{j=0}^{i+h} \gamma_j |x_{i-j}|^{2\nu_j} \le \gamma \sum_{j=0}^{i+h} \gamma_j \epsilon^{2(\nu_j - 1)} x_{i-j}^2, \tag{7.59}$$

$$\begin{aligned}
&2 \sum_{j=0}^{i+h} a_j x_{i-j} g_i(x_{-h}, \dots, x_i) \\
&\quad \le 2 \sum_{j=0}^{i+h} |a_j| \sum_{l=0}^{i+h} \gamma_l |x_{i-l}|^{\nu_l} |x_{i-j}| \\
&\quad \le \sum_{j=0}^{i+h} |a_j| \sum_{l=0}^{i+h} \gamma_l \varepsilon^{\nu_l - 1} \big(x_{i-l}^2 + x_{i-j}^2\big) \le \sum_{j=0}^{i+h} \big(a \gamma_j \varepsilon^{\nu_j - 1} + \mu_1(\varepsilon) |a_j|\big) x_{i-j}^2.
\end{aligned}$$

Therefore,

$$\mathbf{E}_i \Delta V_{1i} \le -x_i^2 + \sum_{j=0}^{i+h} A_j x_{i-j}^2, \tag{7.60}$$

where

$$A_j = \big(a + \mu_1(\varepsilon)\big)|a_j| + \sigma|\sigma_j| + \big(a + \gamma\varepsilon^{\nu_j - 1}\big)\gamma_j\varepsilon^{\nu_j - 1}. \tag{7.61}$$

Choosing the additional functional V_{2i} in the form

$$V_{2i} = \sum_{j=1}^{i+h} x_{i-j}^2 \sum_{l=j}^{\infty} A_l, \tag{7.62}$$

we get

$$\begin{aligned}
\mathbf{E}_i \Delta V_{2i} &= \sum_{j=1}^{i+1+h} x_{i+1-j}^2 \sum_{l=j}^{\infty} A_l - \sum_{j=1}^{i+h} x_{i-j}^2 \sum_{l=j}^{\infty} A_l \\
&= \sum_{j=0}^{i+h} x_{i-j}^2 \sum_{l=j+1}^{\infty} A_l - \sum_{j=1}^{i+h} x_{i-j}^2 \sum_{l=j}^{\infty} A_l \\
&= x_i^2 \sum_{l=1}^{\infty} A_l - \sum_{j=1}^{i+h} A_j x_{i-j}^2.
\end{aligned} \tag{7.63}$$

Via (7.60) and (7.63) for $V_i = V_{1i} + V_{2i}$ we have

$$\mathbf{E}_i \Delta V_i \le \left(\sum_{j=0}^{\infty} A_j - 1 \right) x_i^2$$

or using (7.61)

$$\mathbf{E}_i \Delta V_i \le \big(a^2 + \sigma^2 + 2a\mu_1(\varepsilon) + \gamma\mu_2(\varepsilon) - 1\big) x_i^2.$$

From (7.56) it follows that $\mathbf{E}_i \Delta V_i \le 0$ for small enough ε. It is easy to see that the constructed functional V_i satisfies the conditions $V_i \ge x_i^2$ and $V_0 \le (1+h)\|\varphi\|_0^2$. Thus for the functional V_i the conditions of Theorem 7.3 hold. Therefore, using Remark 7.7 we find that the trivial solution of (7.53) is stable in probability. The theorem is proven.

Let γ be defined in (7.55) and

$$\alpha_l = \left| \sum_{j=l}^{\infty} a_j \right|, \qquad \alpha = \sum_{l=1}^{\infty} \alpha_l, \qquad \beta = \sum_{j=0}^{\infty} a_j. \tag{7.64}$$

□

Theorem 7.5 *If $\gamma < \infty$ and*

$$\beta^2 + 2\alpha|1 - \beta| + \sigma^2 < 1 \tag{7.65}$$

then the trivial solution of (7.53) is stable in probability.

Proof Represent (7.53) in the form

$$x_{i+1}=\beta x_i+\Delta F_i+\sum_{j=0}^{i+h}\sigma_j x_{i-j}\xi_{i+1}+g_i(x_{-h},\dots,x_i) \tag{7.66}$$

with

$$F_i=-\sum_{l=1}^{i+h}x_{i-l}\sum_{j=l}^{\infty}a_j. \tag{7.67}$$

Using the procedure of the construction of the Lyapunov functionals we will construct a Lyapunov functional V_i in the form $V_i=V_{1i}+V_{2i}$, where $V_{1i}=(x_i-F_i)^2$. Via (7.66) we have

$$\begin{aligned}
\mathbf{E}_i\Delta V_{1i}&=\mathbf{E}_i(x_{i+1}-F_{i+1})^2-(x_i-F_i)^2\\
&=\mathbf{E}_i\left(\beta x_i-F_i+\sum_{j=0}^{i+h}\sigma_j x_{i-j}\xi_{i+1}+g_i\right)^2-(x_i-F_i)^2\\
&=\big(\beta^2-1\big)x_i^2+\left(\sum_{j=0}^{i+h}\sigma_j x_{i-j}\right)^2+g_i^2+2(1-\beta)x_iF_i+2\beta x_ig_i-2F_ig_i.
\end{aligned}$$

Using $x_j\in U_\varepsilon$, $j\le i$, (7.58), (7.59) and (7.64) we have

$$2|x_iF_i|\le\sum_{l=1}^{i+h}\alpha_l\big(x_i^2+x_{i-l}^2\big)\le\alpha x_i^2+\sum_{l=1}^{i+h}\alpha_l x_{i-l}^2,$$

$$2|x_ig_i|\le\sum_{j=0}^{i+h}\gamma_j\varepsilon^{\nu_j-1}\big(x_i^2+x_{i-j}^2\big)\le\mu_1(\varepsilon)x_i^2+\sum_{j=0}^{i+h}\gamma_j\varepsilon^{\nu_j-1}x_{i-j}^2,$$

$$\begin{aligned}
2|F_ig_i|&\le2\sum_{j=0}^{i+h}\gamma_j\sum_{l=1}^{i+h}\alpha_l|x_{i-l}||x_{i-j}|^{\nu_j}\le\sum_{j=0}^{i+h}\gamma_j\epsilon^{\nu_j-1}\sum_{l=1}^{i+h}\alpha_l\big(x_{i-l}^2+x_{i-j}^2\big)\\
&\le\alpha\gamma_0\varepsilon^{\nu_0-1}x_i^2+\sum_{l=1}^{i+h}\big(\mu_1(\varepsilon)\alpha_l+\alpha\gamma_l\varepsilon^{\nu_l-1}\big)x_{i-l}^2.
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{E}_i\Delta V_{1i}\le{}&\big[\beta^2+|1-\beta|\alpha+|\beta|\mu_1(\varepsilon)+\sigma|\sigma_0|\\
&+\big(\alpha+|\beta|+\gamma\varepsilon^{\nu_0-1}\big)\gamma_0\varepsilon^{\nu_0-1}-1\big]x_i^2+\sum_{j=1}^{i+h}A_jx_{i-j}^2,
\end{aligned} \tag{7.68}$$

where $\mu_k(\varepsilon)$ is defined in (7.57) and

$$A_j = \big(|1-\beta| + \mu_1(\varepsilon)\big)\alpha_l + \sigma|\sigma_j| + \big(\alpha + |\beta| + \gamma\varepsilon^{\nu_j-1}\big)\gamma_j\varepsilon^{\nu_j-1}. \tag{7.69}$$

Choosing the additional functional V_{2i} again in the form (7.62) and using (7.68) and (7.63) for the functional $V_i = V_{1i} + V_{2i}$ we obtain

$$\mathbf{E}_i\Delta V_i \le \big[\beta^2 + 2\alpha|1-\beta| + \sigma^2 - 1 + 2\big(\alpha+|\beta|\big)\mu_1(\varepsilon) + \gamma\mu_2(\varepsilon)\big]x_i^2.$$

From this and (7.65) it follows that for small enough ε there exists $c_2 > 0$ such that

$$\mathbf{E}_i\Delta V_i \le -c_2x_i^2. \tag{7.70}$$

It means that the constructed functional V_i satisfies condition (7.49). It is easy to see that the condition (7.48) for some $c_1 > 0$ holds too. But this functional V_i does not satisfy condition (7.47). So we cannot use Theorem 7.3 and must find another way for proving it.

Let us consider now the random variable τ, which is defined by (7.50). Using (7.51) and the Chebyshev inequality we get

$$\mathbf{P}\Big\{\sup_{i\in Z}|x_i| > \varepsilon/f_0\Big\} \le \mathbf{P}\big\{|x_{\tau+1}| > \varepsilon/f_0\big\} \le \frac{\mathbf{E}_0x_{\tau+1}^2}{\varepsilon^2}. \tag{7.71}$$

To estimate $\mathbf{E}_0x_{\tau+1}^2$ suppose that $0\le k\le i\le\tau$. Then from (7.70) it follows that

$$\mathbf{E}_k\sum_{j=k}^{i}\mathbf{E}_j\Delta V_j = \mathbf{E}_kV_{i+1} - V_k \le -c_2\sum_{j=k}^{i}x_j^2 \le -c_2x_k^2 \le 0.$$

From this we have

$$V_k \ge c_2x_k^2, \quad 0\le k\le\tau, \tag{7.72}$$

$$\mathbf{E}_0V_{i+1} \le V_0, \quad 0\le i\le\tau. \tag{7.73}$$

It is easy to see that

$$V_{\tau+1} \ge (x_{\tau+1} - F_{\tau+1})^2 \ge x_{\tau+1}^2 - 2x_{\tau+1}F_{\tau+1}. \tag{7.74}$$

From (7.67) and (7.64) it follows that

$$\begin{aligned}2x_{\tau+1}F_{\tau+1} &\le \sum_{l=1}^{\tau+1+h}\alpha_l\big(x_{\tau+1}^2 + x_{\tau+1-l}^2\big) \le \alpha x_{\tau+1}^2 + \sum_{l=1}^{\tau+1+h}\alpha_lx_{\tau+1-l}^2\\ &= \alpha x_{\tau+1}^2 + \sum_{l=1}^{\tau}\alpha_lx_{\tau+1-l}^2 + \sum_{l=\tau+1}^{\tau+1+h}\alpha_lx_{\tau+1-l}^2.\end{aligned}$$

Using (7.72), (7.46) and (7.64) we get

$$2x_{\tau+1}F_{\tau+1} \le \alpha x_{\tau+1}^2 + \sum_{l=1}^{\tau} \frac{\alpha_l}{c_2} V_{\tau+1-l} + \alpha\delta^2. \tag{7.75}$$

From (7.65) it follows that $|\beta| < 1$. Thus,

$$\alpha < \frac{1-\beta^2-\sigma^2}{2(1-\beta)} \le \frac{1+\beta}{2} < 1.$$

Substituting (7.75) into (7.74) we get

$$x_{\tau+1}^2 \le \frac{1}{1-\alpha}\left(\alpha\delta^2 + \sum_{l=1}^{\tau} \frac{\alpha_l}{c_2} V_{\tau+1-l} + V_{\tau+1}\right).$$

Calculating $\mathbf{E}_0$ by virtue of (7.73), (7.64) and (7.48), which holds, as was noted above, and (7.46), we get

$$\mathbf{E}_0 x_{\tau+1}^2 \le \frac{1}{1-\alpha}\left(\alpha\delta^2 + \left(\frac{\alpha}{c_2}+1\right)V_0\right) < C\delta^2,$$

where

$$C = \frac{1}{1-\alpha}\left(\alpha + \left(\frac{\alpha}{c_2}+1\right)c_1\right).$$

From this and (7.71) by $\delta = \varepsilon\sqrt{\varepsilon_1/C}$, (7.45) follows. Therefore, the trivial solution of (7.53) is stable in probability. The theorem is proven. □

Remark 7.8 Note that the condition (7.65) can be written in the form

$$\sigma^2 < (1-\beta)(1+\beta-2\alpha), \quad |\beta| < 1.$$

Remark 7.9 It is easy to see that if in (7.53) $g_i = 0$, then the functionals which were constructed in Theorems 7.3 and 7.4 satisfy the conditions of Theorem 1.1. It means that conditions (7.56) and (7.65) are sufficient for asymptotic mean square stability of the trivial solution of the linear part of (7.53). Thus, it is shown that the investigation of stability in probability of nonlinear stochastic difference equations with an order of nonlinearity higher than one can be reduced to the investigation of asymptotic mean square stability of the linear part of these equations.

Example 7.4 Consider the nonlinear difference equation

$$x_{i+1} = \frac{ax_i}{1+b\sin(x_{i-k})} + \sigma x_{i-m}\xi_{i+1}, \tag{7.76}$$

where k and m are nonnegative integers and

$$|b| < 1. \tag{7.77}$$

Using inequalities (7.77) and $|\sin(x)| \le 1$ for the functional $V_i = x_i^2$ we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) \\
&= \mathbf{E}\left[\left(\frac{ax_i}{1+b\sin(x_{i-k})} + \sigma x_{i-m}\xi_{i+1}\right)^2 - x_i^2\right] \\
&= \left(\frac{a^2}{(1+b\sin(x_{i-k}))^2} - 1\right)\mathbf{E}x_i^2 + \sigma^2\mathbf{E}x_{i-m}^2 \\
&\le \left(\frac{a^2}{(1-|b|)^2} - 1\right)\mathbf{E}x_i^2 + \sigma^2\mathbf{E}x_{i-m}^2.
\end{aligned}$$

From Theorem 1.2 it follows that the inequality

$$\frac{a^2}{(1-|b|)^2} + \sigma^2 < 1 \tag{7.78}$$

is a sufficient condition for the asymptotic mean square stability of the trivial solution of (7.76).

Let us obtain a sufficient condition for stability in probability of the trivial solution of (7.76). Rewrite (7.76) in the form

$$x_{i+1} = ax_i + g(x_{i-k}, x_i) + \sigma x_{i-m}\xi_{i+1}, \tag{7.79}$$

where

$$g(x_{i-k}, x_i) = -\frac{abx_i \sin(x_{i-k})}{1+b\sin(x_{i-k})}.$$

Using the inequality $|\sin(x)| \le |x|$ in the numerator of the fraction and the inequalities $|\sin(x)| \le 1$ and (7.77) in the denominator we have

$$\big|g(x_{i-k}, x_i)\big| \le \frac{|abx_i x_{i-k}|}{1-|b|} \le \frac{|ab|}{2(1-|b|)}\big(x_i^2 + x_{i-k}^2\big).$$

It means that the function $g(x_{i-k}, x_i)$ satisfies the condition (7.54). Therefore, the condition

$$a^2 + \sigma^2 < 1, \tag{7.80}$$

which follows from (7.78) by $b = 0$, is a sufficient condition for asymptotic mean square stability of the linear part of (7.79) and, as follows from Theorem 7.4, is a sufficient condition for the stability in probability of the trivial solution of (7.76) for all b satisfying condition (7.77).

It is easy to see that the condition (7.80) for stability in probability is weaker than condition (7.78) for asymptotic mean square stability.

7.3 Fractional Difference Equations

There is a very large interest in studying the behavior of solutions of the special type of nonlinear difference equations, so called, fractional difference equations [1, 2, 10, 17–19, 37–39, 43–48, 55, 64, 65, 85, 86, 94, 100, 108, 147, 155–158, 169, 170, 193, 199, 257, 258, 260–262, 271, 273, 274]. Here the conditions for stability in probability for fractional difference equations with stochastic perturbations are obtained. Numerous graphical illustrations of stability regions and the trajectories of solutions are plotted.

7.3.1 Equilibrium Points

Consider the fractional difference equation

$$x_{i+1}=\frac{\mu+\sum_{j=0}^{k}a_jx_{i-j}}{\lambda+\sum_{j=0}^{k}b_jx_{i-j}},\quad i\in Z, \tag{7.81}$$

with the initial condition

$$x_j=\varphi_j,\quad j\in Z_0=\{-k,-k+1,\dots,0\}. \tag{7.82}$$

Here μ, λ, a_j, b_j, $j=0,\dots,k$, are known constants. Equation (7.81) generalizes a lot of different particular cases that were considered in [1, 2, 10, 17–19, 38, 39, 55, 64, 85, 86, 94, 100, 108, 147, 258, 262, 273].

Put

$$\begin{aligned}&A_j=\sum_{l=j}^{k}a_l,\qquad B_j=\sum_{l=j}^{k}b_l,\quad j=0,1,\dots,k,\\&A=A_0,\qquad B=B_0,\end{aligned} \tag{7.83}$$

and suppose that (7.81) has some point of equilibrium $\hat{x}$ (not necessary a positive one). Then by assumption

$$\lambda+B\hat{x}\neq 0 \tag{7.84}$$

the equilibrium point $\hat{x}$ is defined by the algebraic equation

$$\hat{x}=\frac{\mu+A\hat{x}}{\lambda+B\hat{x}}. \tag{7.85}$$

By the condition (7.84), (7.85) can be transformed into the form

$$B\hat{x}^2-(A-\lambda)\hat{x}-\mu=0. \tag{7.86}$$

It is clear that if

$$(A-\lambda)^2+4B\mu>0 \tag{7.87}$$

then (7.81) has two points of equilibrium:

$$\hat{x}_1=\frac{A-\lambda+\sqrt{(A-\lambda)^2+4B\mu}}{2B} \tag{7.88}$$

and

$$\hat{x}_2=\frac{A-\lambda-\sqrt{(A-\lambda)^2+4B\mu}}{2B}, \tag{7.89}$$

and if

$$(A-\lambda)^2+4B\mu=0 \tag{7.90}$$

then (7.81) has only one point of equilibrium,

$$\hat{x}=\frac{A-\lambda}{2B}. \tag{7.91}$$

And finally if

$$(A-\lambda)^2+4B\mu<0 \tag{7.92}$$

then (7.81) has no equilibrium points.

Remark 7.10 Consider the case $\mu=0$, $B\neq 0$. From (7.85) we obtain the following. If $\lambda\neq 0$ and $A\neq\lambda$, then (7.81) has two points of equilibrium:

$$\hat{x}_1=\frac{A-\lambda}{B}, \qquad \hat{x}_2=0. \tag{7.93}$$

If $\lambda\neq 0$ and $A=\lambda$ then (7.81) has only one point of equilibrium: $\hat{x}=0$. If $\lambda=0$ then (7.81) has only one point of equilibrium: $\hat{x}=AB^{-1}$.

Remark 7.11 Consider the case $\mu=B=0$, $\lambda\neq 0$. If $A\neq\lambda$ then (7.81) has only one point of equilibrium: $\hat{x}=0$. If $A=\lambda$ then each solution $\hat{x}=\text{const}$ is an equilibrium point of (7.81).

7.3.2 Stochastic Perturbations, Centering and Linearization

Let $\{\Omega,\mathfrak{F},\mathbf{P}\}$ be a probability space and $\{\mathfrak{F}_i, i\in Z\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, ξ_i, $i\in Z$, be a sequence of $\mathfrak{F}_i$-adapted mutually independent random variables such that $\mathbf{E}\xi_i=0$, $\mathbf{E}\xi_i^2=1$.

As was first proposed in [24, 235] and used later in [16, 30, 41], we will suppose that (7.81) is exposed to stochastic perturbations ξ_i which are directly proportional to the deviation of the state x_i of system (7.81) from the equilibrium point $\hat{x}$. So (7.81) takes the form

$$x_{i+1} = \frac{\mu + \sum_{j=0}^{k} a_j x_{i-j}}{\lambda + \sum_{j=0}^{k} b_j x_{i-j}} + \sigma(x_i - \hat{x})\xi_{i+1}. \tag{7.94}$$

Note that the equilibrium point $\hat{x}$ of (7.81) is also the equilibrium point of (7.94)

Putting $y_i = x_i - \hat{x}$ we will center (7.94) in the neighborhood of the point of equilibrium $\hat{x}$. From (7.94) and (7.85) it follows that

$$y_{i+1} = \frac{\sum_{j=0}^{k}(a_j - b_j\hat{x})y_{i-j}}{\lambda + B\hat{x} + \sum_{j=0}^{k} b_j y_{i-j}} + \sigma y_i \xi_{i+1}. \tag{7.95}$$

It is clear that stability of the trivial solution of (7.95) is equivalent to stability of the equilibrium point $\hat{x}$ of (7.94).

Together with nonlinear equation (7.95) we will consider its linear part

$$z_{i+1} = \sum_{j=0}^{k} \alpha_j z_{i-j} + \sigma z_i \xi_{i+1}, \quad \alpha_j = \frac{a_j - b_j\hat{x}}{\lambda + B\hat{x}}. \tag{7.96}$$

From (3.3) for (7.96) it follows that if

$$\sum_{j=0}^{k} |\alpha_j| < \sqrt{1 - \sigma^2} \tag{7.97}$$

then the trivial solution of (7.96) is asymptotically mean square stable.

Put

$$\alpha = \sum_{j=1}^{k} \left| \sum_{l=j}^{k} \alpha_l \right|, \qquad \beta = \sum_{j=0}^{k} \alpha_j. \tag{7.98}$$

If

$$\beta^2 + 2\alpha|1 - \beta| + \sigma^2 < 1 \tag{7.99}$$

then the trivial solution of (7.96) is asymptotically mean square stable.

Let U and $\mathfrak{A}$ be two square matrices of dimension $k+1$ such that $U = \|u_{ij}\|$ has all zero elements except for $u_{k+1,k+1} = 1$ and

$$\mathfrak{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \alpha_k & \alpha_{k-1} & \alpha_{k-2} & \dots & \alpha_1 & \alpha_0 \end{pmatrix}.$$

Let the matrix equation

$$\mathfrak{A}' D \mathfrak{A} - D = -U \tag{7.100}$$

have a positively semidefinite solution D with $d_{k+1,k+1} > 0$. Then the trivial solution of (7.45) is asymptotically mean square stable if and only if

$$\sigma^2 d_{k+1,k+1} < 1. \tag{7.101}$$

For example, for $k = 1$ the condition (7.101) takes the form

$$|\alpha_1| < 1, \quad |\alpha_0| < 1 - \alpha_1, \tag{7.102}$$

$$\sigma^2 < d_{22}^{-1} = 1 - \alpha_1^2 - \alpha_0^2 \frac{1 + \alpha_1}{1 - \alpha_1}. \tag{7.103}$$

If, in particular, $\sigma = 0$ then condition (7.102) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (7.96) for $k = 1$.

Remark 7.12 Put $\sigma = 0$. If $\beta = 1$, then the trivial solution of (7.96) can be stable (for example, $z_{i+1} = z_i$ or $z_{i+1} = 0.5(z_i + z_{i-1})$), unstable (for example, $z_{i+1} = 2z_i - z_{i-1}$) but cannot be asymptotically stable. In fact, it is easy to see that if $\beta \geq 1$ (in particular, $\beta = 1$) then the sufficient conditions (7.97) and (7.99) do not hold. Moreover, the necessary and sufficient (for $k = 1$) condition (7.102) does not hold too since by the condition (7.102) we obtain the contradiction $1 \leq \beta = \alpha_0 + \alpha_1 \leq |\alpha_0| + \alpha_1 < 1$.

Remark 7.13 As follows from Remark 7.9, the conditions (7.97), (7.99) and (7.101) for asymptotic mean square stability of the trivial solution of (7.96) at the same time are conditions for stability in probability of the equilibrium point of (7.94).

7.3.3 Stability of Equilibrium Points

From conditions (7.97) and (7.99) it follows that $|\beta| < 1$. Let us check if this condition can be true for each equilibrium point.

Suppose at first that the condition (7.87) holds. Then (7.94) has two points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ defined by (7.88) and (7.89), respectively. Putting $S = \sqrt{(A-\lambda)^2 + 4B\mu}$ via (7.98), (7.96) and (7.83) we find that the corresponding β_1 and β_2 are

$$\begin{aligned} \beta_1 &= \frac{A - B\hat{x}_1}{\lambda + B\hat{x}_1} = \frac{A - \frac{1}{2}(A - \lambda + S)}{\lambda + \frac{1}{2}(A - \lambda + S)} = \frac{A + \lambda - S}{A + \lambda + S}, \\ \beta_2 &= \frac{A - B\hat{x}_2}{\lambda + B\hat{x}_2} = \frac{A - \frac{1}{2}(A - \lambda - S)}{\lambda + \frac{1}{2}(A - \lambda - S)} = \frac{A + \lambda + S}{A + \lambda - S}. \end{aligned} \tag{7.104}$$

So $\beta_1\beta_2 = 1$. It means that the condition $|\beta| < 1$ holds only for one of the two equilibrium points $\hat{x}_1$ and $\hat{x}_2$. Namely, if $A+\lambda > 0$ then $|\beta_1| < 1$, if $A+\lambda < 0$ then $|\beta_2| < 1$, if $A+\lambda = 0$ then $\beta_1 = \beta_2 = -1$. In particular, if $\mu = 0$, then via Remark 7.10 and (7.96) we have $\beta_1 = \lambda A^{-1}$, $\beta_2 = \lambda^{-1}A$. Therefore, $|\beta_1| < 1$ if $|\lambda| < |A|$, $|\beta_2| < 1$ if $|\lambda| > |A|$, $|\beta_1| = |\beta_2| = 1$ if $|\lambda| = |A|$.

So, via Remark 7.12 we find that the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ can be stable concurrently only if the corresponding β_1 and β_2 are negative concurrently.

Suppose now that the condition (7.90) holds. Then (7.94) has only one point of equilibrium (7.91). From (7.98), (7.96), (7.83) and (7.91) it follows that corresponding β equals

$$\beta = \frac{A - B\hat{x}}{\lambda + B\hat{x}} = \frac{A - \frac{1}{2}(A-\lambda)}{\lambda + \frac{1}{2}(A-\lambda)} = \frac{A+\lambda}{\lambda + A} = 1.$$

As follows from Remark 7.12, this point of equilibrium cannot be asymptotically stable.

Corollary 7.1 *Let $\hat{x}$ be an equilibrium point of* (7.94) *such that*

$$\sum_{j=0}^{k} |a_j - b_j\hat{x}| < |\lambda + B\hat{x}|\sqrt{1-\sigma^2}, \quad \sigma^2 < 1. \tag{7.105}$$

Then the equilibrium point $\hat{x}$ is stable in probability.

The proof follows from (7.96), (7.97) and Remark 7.13.

Corollary 7.2 *Let $\hat{x}$ be an equilibrium point of* (7.94) *such that*

$$|A - B\hat{x}| < |\lambda + B\hat{x}|, \tag{7.106}$$

$$2\sum_{j=1}^{k} |A_j - B_j\hat{x}| < |\lambda + A| - \sigma^2 \frac{(\lambda + B\hat{x})^2}{|\lambda - A + 2B\hat{x}|}. \tag{7.107}$$

Then the equilibrium point $\hat{x}$ is stable in probability.

Proof Via (7.83), (7.96) and (7.98) we have

$$\alpha = |\lambda + B\hat{x}|^{-1} \sum_{j=1}^{k} |A_j - B_j\hat{x}|, \qquad \beta = \frac{A - B\hat{x}}{\lambda + B\hat{x}}.$$

We rewrite equation (7.99) in the form

$$2\alpha < 1 + \beta - \frac{\sigma^2}{1-\beta}, \quad |\beta| < 1,$$

and we show that it follows from (7.106) and (7.107). In fact, from (7.106) it follows that $|\beta| < 1$. Via $|\beta| < 1$ we have

$$1+\beta = 1+\frac{A-B\hat{x}}{\lambda+B\hat{x}} = \frac{\lambda+A}{\lambda+B\hat{x}} > 0$$

and

$$1-\beta = 1-\frac{A-B\hat{x}}{\lambda+B\hat{x}} = \frac{\lambda-A+2B\hat{x}}{\lambda+B\hat{x}} > 0.$$

So, via (7.107)

$$\begin{aligned} 2\alpha &= 2\sum_{j=1}^{k}\frac{|A_j-B_j\hat{x}|}{|\lambda+B\hat{x}|} < \frac{|\lambda+A|}{|\lambda+B\hat{x}|} - \sigma^2\frac{|\lambda+B\hat{x}|}{|\lambda-A+2B\hat{x}|} \\ &= 1+\beta-\frac{\sigma^2}{1-\beta}. \end{aligned}$$

It means that the condition (7.99) holds. Via Remark 7.13 the proof is completed. □

Corollary 7.3 *An equilibrium point $\hat{x}$ of the equation*

$$x_{i+1} = \frac{\mu+a_0x_i+a_1x_{i-1}}{\lambda+b_0x_i+b_1x_{i-1}} + \sigma(x_i-\hat{x})\xi_{i+1} \tag{7.108}$$

is stable in probability if and only if

$$\begin{aligned} &|a_1-b_1\hat{x}| < |\lambda+B\hat{x}|, \\ &|a_0-b_0\hat{x}| < \big(\lambda-a_1+(b_0+2b_1)\hat{x}\big)\,\mathrm{sign}(\lambda+B\hat{x}), \end{aligned} \tag{7.109}$$

$$\sigma^2 < \frac{(\lambda+a_1+b_0\hat{x})(\lambda+a_0-a_1+2b_1\hat{x})(\lambda-A+2B\hat{x})}{(\lambda-a_1+(b_0+2b_1)\hat{x})(\lambda+B\hat{x})^2}. \tag{7.110}$$

The proof follows from (7.96), (7.102) and (7.103).

7.3.4 Examples

Below some examples are considered as an application of the results obtained.

Example 7.5 Consider (7.108) with $a_0 = 2.9$, $a_1 = 0.1$, $b_0 = b_1 = 0.5$. From (7.83) and (7.87)–(7.89) it follows that $A = 3$, $B = 1$ and, for any fixed μ and λ such that $\mu > -\frac{1}{4}(3-\lambda)^2$, (7.108) has two points of equilibrium:

$$\hat{x}_1 = \frac{1}{2}\Big(3-\lambda+\sqrt{(3-\lambda)^2+4\mu}\Big), \qquad \hat{x}_2 = \frac{1}{2}\Big(3-\lambda-\sqrt{(3-\lambda)^2+4\mu}\Big).$$

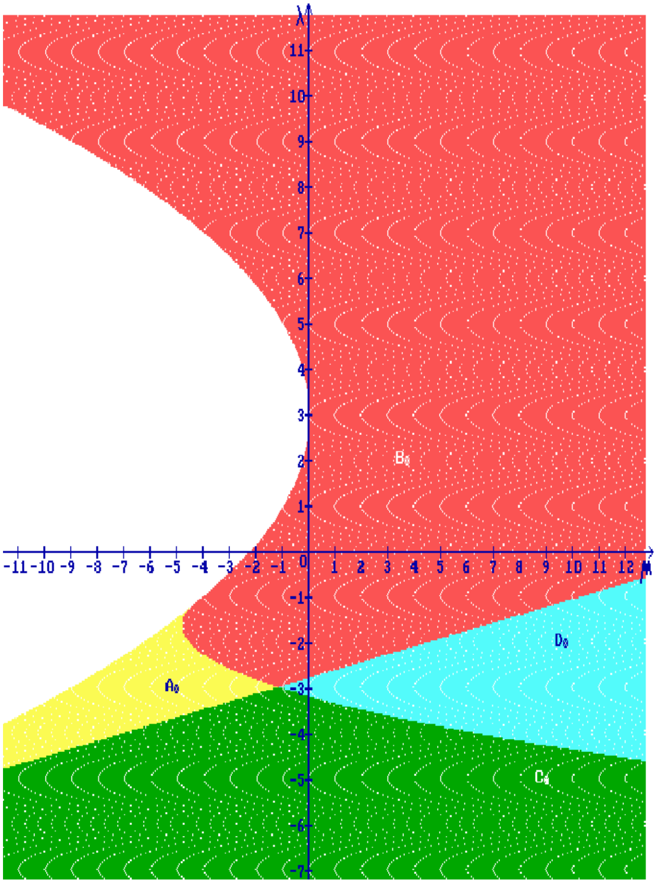


Fig. 7.4 Stability regions, $\sigma^2 = 0$

In Fig. 7.4 the region where the points of equilibrium are absent (white region), the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are though being unstable (yellow region), the region where the point of equilibrium $\hat{x}_1$ is stable only (red region), the region where the point of equilibrium $\hat{x}_2$ is stable only (green region) and the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are stable (cyan region) are shown in the space of (μ, λ). All regions are obtained via the conditions (7.109) for $\sigma^2 = 0$. In Figs. 7.5 and 7.6 one can see similar regions for $\sigma^2 = 0.3$ and $\sigma^2 = 0.8$, in accordance to those obtained via the conditions (7.109) and (7.110). In Fig. 7.7 it is shown that sufficient conditions (7.105), (7.106) and (7.107) are close enough to the necessary and sufficient conditions (7.109) and (7.110): inside of the region where the point of equilibrium $\hat{x}_1$ is stable (red region) one can see the regions of stability of the point of equilibrium $\hat{x}_1$ that were obtained by condition (7.105) (grey and green regions) and by the conditions (7.106) and (7.107) (cyan and green regions). Stability regions obtained via both sufficient conditions of stability (7.105),

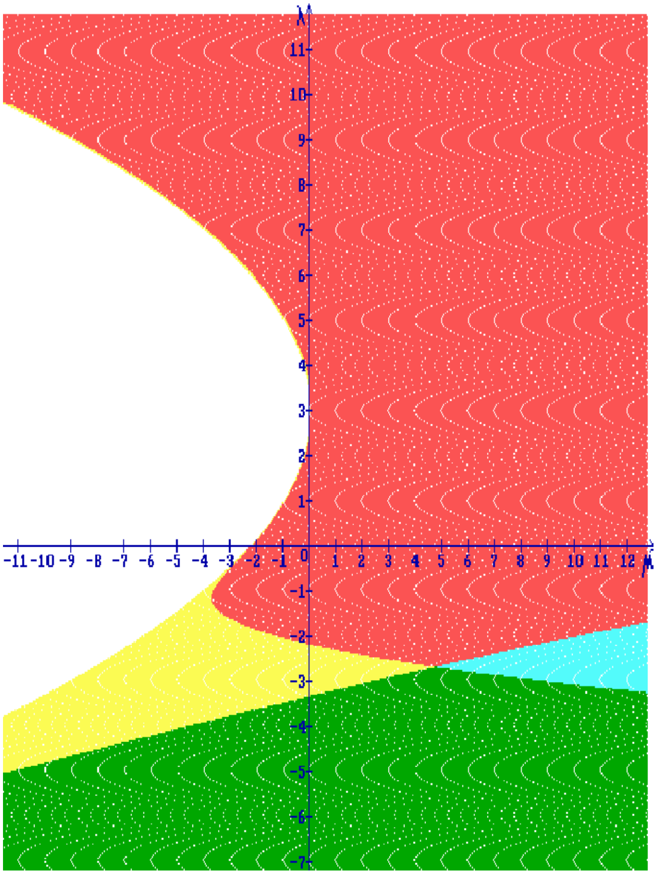


Fig. 7.5 Stability regions, $\sigma^2 = 0.3$

(7.106) and (7.107) give together almost the whole stability region obtained via the necessary and sufficient stability conditions (7.109) and (7.110).

Consider now the behavior of the solutions of (7.108) with $\sigma = 0$ in points A, B, C and D of the space of (μ, λ) (Fig. 7.4). In the point A with $\mu = -5$, $\lambda = -3$, both equilibrium points $\hat{x}_1 = 5$ and $\hat{x}_2 = 1$ are unstable. In Fig. 7.8 two trajectories of solutions of (7.108) are shown with the initial conditions $x_{-1} = 5$, $x_0 = 4.95$ and $x_{-1} = 0.999$, $x_0 = 1.0001$. In Fig. 7.9 two trajectories of solutions of (7.108) with the initial conditions $x_{-1} = -3$, $x_0 = 13$ and $x_{-1} = -1.5$, $x_0 = -1.500001$ are shown in point B with $\mu = 3.75$, $\lambda = 2$. One can see that the equilibrium point $\hat{x}_1 = 2.5$ is stable and the equilibrium point $\hat{x}_2 = -1.5$ is unstable. In point C with $\mu = 9$, $\lambda = -5$ the equilibrium point $\hat{x}_1 = 9$ is unstable and the equilibrium point $\hat{x}_2 = -1$ is stable. Two corresponding trajectories of solutions are shown in Fig. 7.10 with the initial conditions $x_{-1} = 7$, $x_0 = 10$ and $x_{-1} = -8$, $x_0 = 8$. In point D with $\mu = 9.75$, $\lambda = -2$ both equilibrium points $\hat{x}_1 = 6.5$ and $\hat{x}_2 = -1.5$ are stable. Two corresponding trajectories of solutions are shown in Fig. 7.11 with the initial

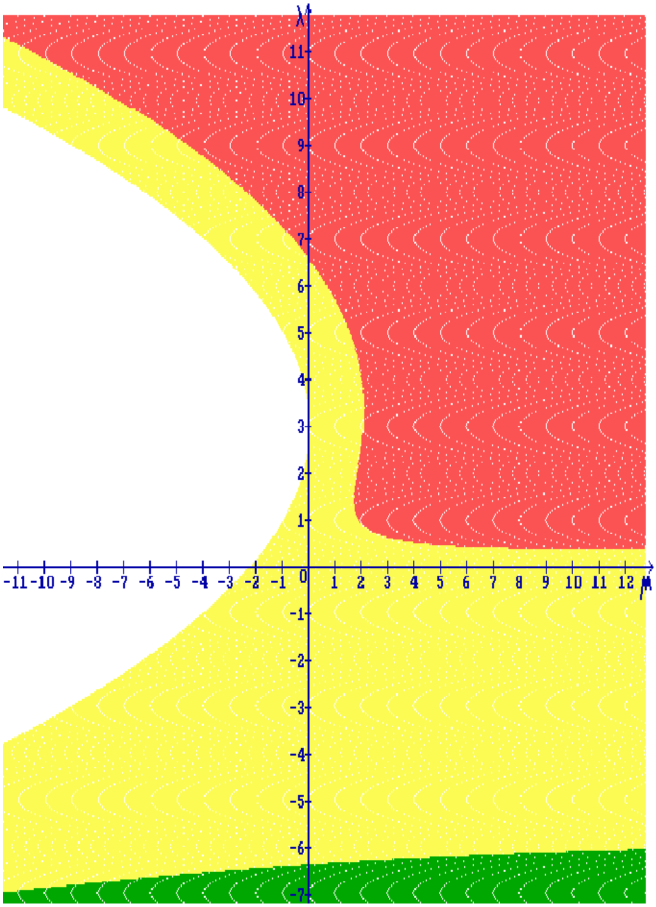


Fig. 7.6 Stability regions, $\sigma^2 = 0.8$

conditions $x_{-1} = 2$, $x_0 = 12$ and $x_{-1} = -8$, $x_0 = 8$. As was noted above in this case the corresponding β_1 and β_2 are negative: $\beta_1 = -7/9$ and $\beta_2 = -9/7$.

Example 7.6 Consider the difference equation

$$x_{i+1} = p + q\frac{x_{i-m}}{x_{i-r}} + \sigma(x_i - \hat{x})\xi_{i+1}. \tag{7.111}$$

Different particular cases of this equation were considered in [2, 10, 18, 19, 55, 94, 100, 273].

Equation (7.111) is a particular case of (7.94) with

$$\begin{aligned} a_r &= p, & a_m &= q, & a_j &= 0 \quad \text{if } j \neq r \text{ and } j \neq m, \\ \mu = \lambda &= 0, & b_r &= 1, & b_j &= 0 \quad \text{if } j \neq r,\ \hat{x} = p + q. \end{aligned}$$

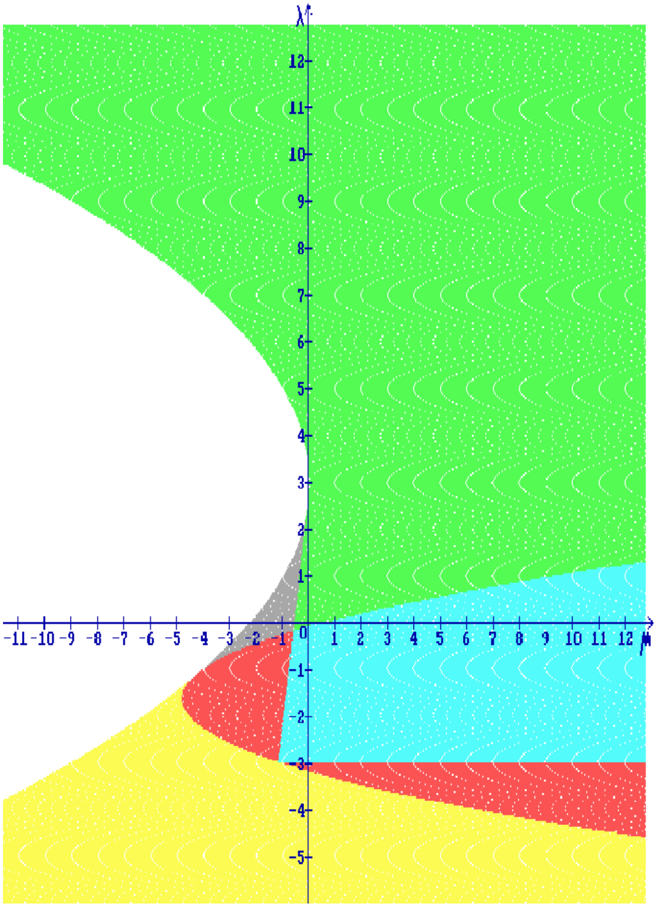


Fig. 7.7 Stability regions, $\sigma^2 = 0$

Suppose first that $p+q \neq 0$ and consider two cases: (1) $m > r \geq 0$, (2) $r > m \geq 0$. In the first case

$$A_j = \begin{cases} p+q, & j = 0, \ldots, r, \\ q, & j = r+1, \ldots, m, \end{cases} \qquad B_j = \begin{cases} 1, & j = 0, \ldots, r, \\ 0, & j = r+1, \ldots, m. \end{cases}$$

In the second case

$$A_j = \begin{cases} p+q, & j = 0, \ldots, m, \\ p, & j = m+1, \ldots, r, \end{cases} \qquad B_j = 1, \quad j = 0, \ldots, r.$$

In both these cases Corollary 7.1 gives a sufficient condition for stability in probability for the equilibrium point $\hat{x} = p+q$ in the form $2|q| < \sqrt{1-\sigma^2}|p+q|$ or

$$p \in \big(-\infty, -q - \theta|q|\big) \cup \big(-q + \theta|q|, \infty\big) \tag{7.112}$$

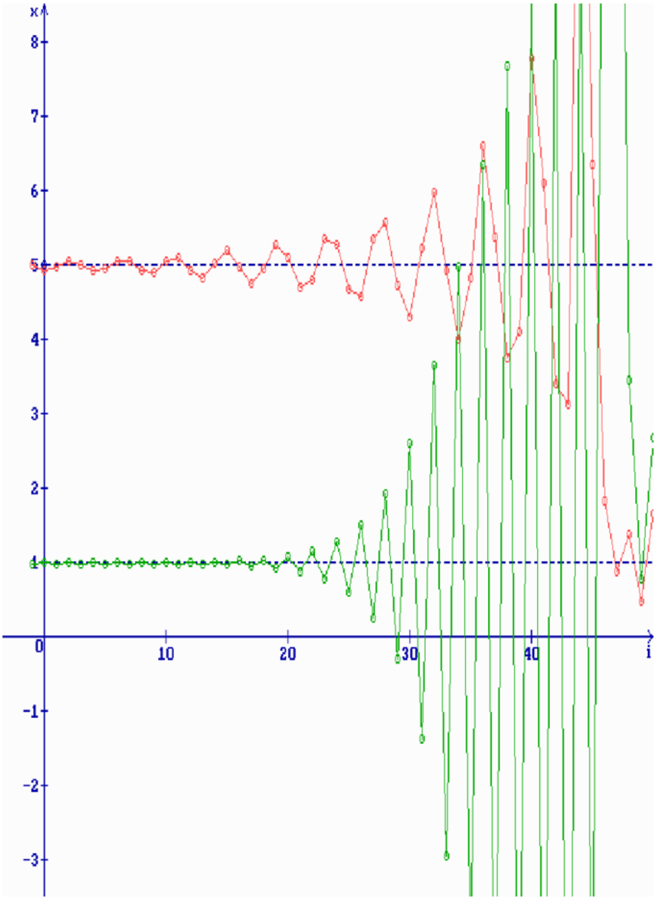


Fig. 7.8 Unstable equilibrium points $\hat{x}_1 = 5$ and $\hat{x}_2 = 1$ for $\mu = -5$, $\lambda = -3$

with

$$\theta = \theta_1 = \frac{2}{\sqrt{1-\sigma^2}}. \tag{7.113}$$

Corollary 7.2 in both cases gives a sufficient condition for stability in probability for the equilibrium point $\hat{x} = p + q$ in the form $2|q||m - r| < (1 - \sigma^2)|p + q|$ or (7.112) with

$$\theta = \theta_2 = \frac{2|m-r|}{1-\sigma^2}. \tag{7.114}$$

Since $\theta_2 > \theta_1$, condition (7.112) with (7.113) is better than (7.112) and (7.114).

In the case $m = 1$, $r = 0$ Corollary 7.3 gives a stability condition in the form

$$|q| < |p+q|, \qquad |q| < p\,\mathrm{sign}(p+q), \qquad \sigma^2 < \frac{(p+2q)(p-q)}{p(p+q)}$$

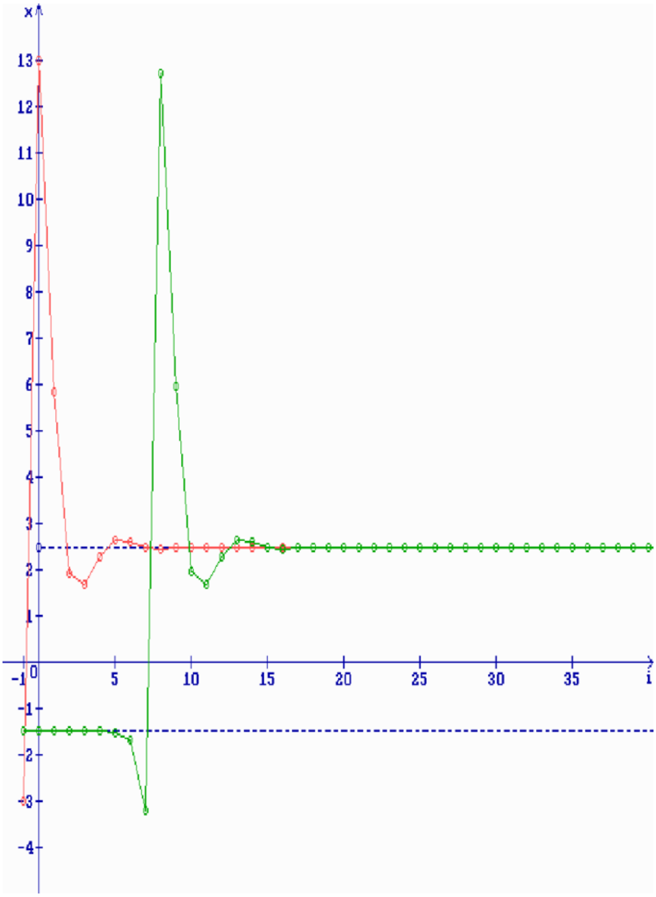


Fig. 7.9 Stable equilibrium point $\hat{x}_1 = 2.5$ and unstable $\hat{x}_2 = -1.5$ for $\mu = 3.75$, $\lambda = 2$

or

$$p \in \left(-\infty, \frac{1}{2}\left(-q-\theta|q|\right)\right) \cup \left(\frac{1}{2}\left(-q+\theta|q|\right),\right), \quad \theta = \sqrt{\frac{9-\sigma^2}{1-\sigma^2}}. \tag{7.115}$$

In particular, from (7.115) it follows that, for $q = 1$, $\sigma = 0$ (this case was considered in [10, 100]), the equilibrium point $\hat{x} = p + 1$ is stable if and only if $p \in (-\infty, -2) \cup (1, \infty)$. Note that in [10] for this case the condition $p > 1$ only was obtained.

In Fig. 7.12 four trajectories of solutions of (7.111) in the case $m = 1$, $r = 0$, $\sigma = 0$, $q = 1$ are shown: (1) $p = 2$, $\hat{x} = 3$, $x_{-1} = 4$, $x_0 = 1$ (red line, stable solution); (2) $p = 0.93$, $\hat{x} = 1.93$, $x_{-1} = 2.1$, $x_0 = 1.7$ (brown line, unstable solution); (3) $p = -1.9$, $\hat{x} = -0.9$, $x_{-1} = -0.89$, $x_0 = -0.94$ (blue line, unstable solution); (4) $p = -2.8$, $\hat{x} = -1.8$, $x_{-1} = -4$, $x_0 = 3$ (green line, stable solution).

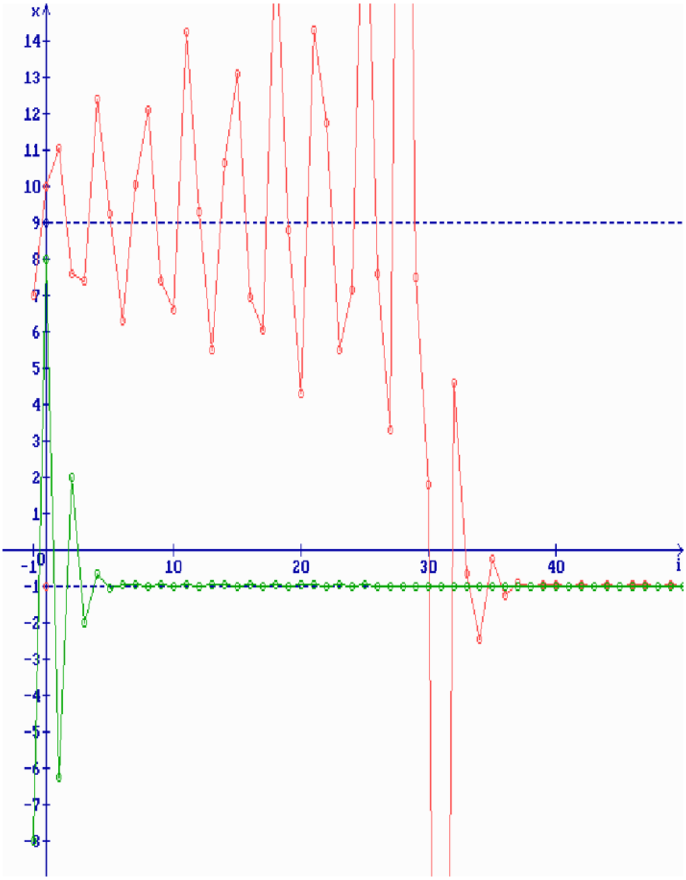


Fig. 7.10 Unstable equilibrium point $\hat{x}_1 = 9$ and stable $\hat{x}_2 = -1$ for $\mu = 9$, $\lambda = -5$

In the case $r = 1$, $m = 0$ Corollary 7.3 gives a stability condition in the form

$$|q| < |p+q|, \ |q| < (p+2q)\,\mathrm{sign}(p+q), \quad \sigma^2 < \frac{p(p+3q)}{(p+q)(p+2q)}$$

or

$$p \in \left(-\infty, \frac{1}{2}\big(-3q - \theta|q|\big)\right) \cup \left(\frac{1}{2}\big(-3q + \theta|q|\big), \infty\right), \quad \theta = \sqrt{\frac{9-\sigma^2}{1-\sigma^2}}. \quad (7.116)$$

For example, from (7.116) it follows that for $q = -1$, $\sigma = 0$ (this case was considered in [94, 273]) the equilibrium point $\hat{x} = p - 1$ is stable if and only if $p \in (-\infty, 0) \cup (3, \infty)$. In Fig. 7.13 four trajectories of solutions of (7.111) in the case $r = 1$, $m = 0$, $\sigma = 0$, $q = -1$ are shown: (1) $p = 3.5$, $\hat{x} = 2.5$, $x_{-1} = 3.5$, $x_0 = 1.5$ (red line, stable solution); (2) $p = 2.2$, $\hat{x} = 1.2$, $x_{-1} = 1.2$, $x_0 = 1.2001$

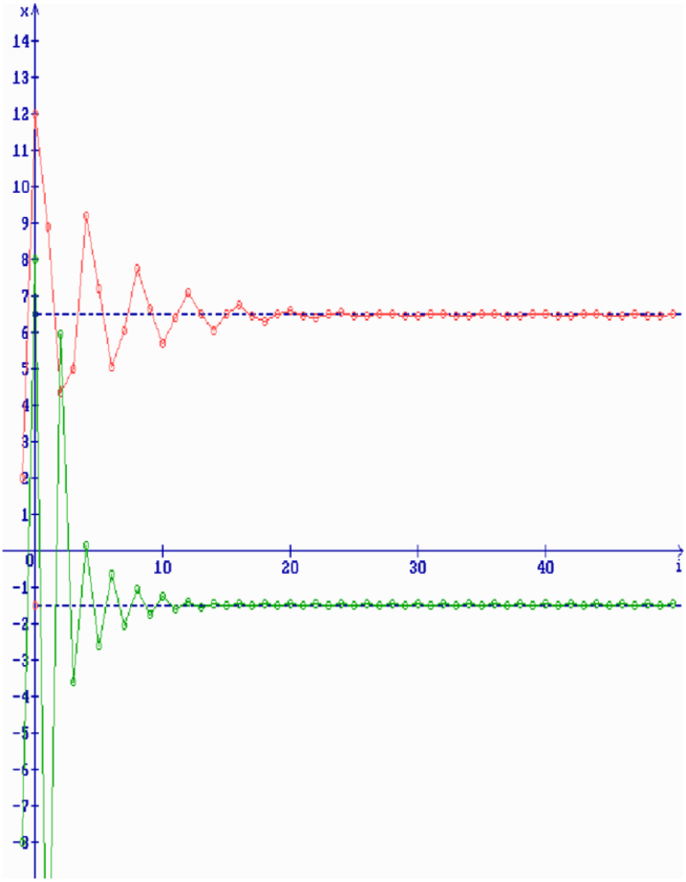


Fig. 7.11 Stable equilibrium points $\hat{x}_1 = 6.5$ and $\hat{x}_2 = -1.5$ for $\mu = 9.75$, $\lambda = -2$

(brown line, unstable solution); (3) $p = 0.3$, $\hat{x} = -0.7$, $x_{-1} = -0.7$, $x_0 = -0.705$ (blue line, unstable solution); (4) $p = -0.2$, $\hat{x} = -1.2$, $x_{-1} = -2$, $x_0 = -0.4$ (green line, stable solution).

Via a simulation of a sequence of mutually independent random variables ξ_i, $i \in Z$, consider the behavior of the equilibrium point by stochastic perturbations. In Fig. 7.14 1000 trajectories are shown for $p = 4$, $q = -1$, $\sigma = 0.5$, $x_{-1} = 3.5$, $x_0 = 2.5$. In this case the stability condition (7.116) holds ($4 \in (-\infty, -0.21) \cup (3.21, \infty)$) and therefore the equilibrium point $\hat{x} = 3$ is stable: all trajectories go to $\hat{x}$. Putting $\sigma = 0.9$, we find that the stability condition (7.116) does not hold ($4 \notin (-\infty, -1.78) \cup (4.78, \infty)$). Therefore, the equilibrium point $\hat{x} = 3$ is unstable: in Fig. 7.15 one can see that 1000 trajectories fill the whole space.

Note also that if $p + q$ goes to zero all obtained stability conditions are violated. Therefore, by the conditions $p + q = 0$, the equilibrium point is unstable.

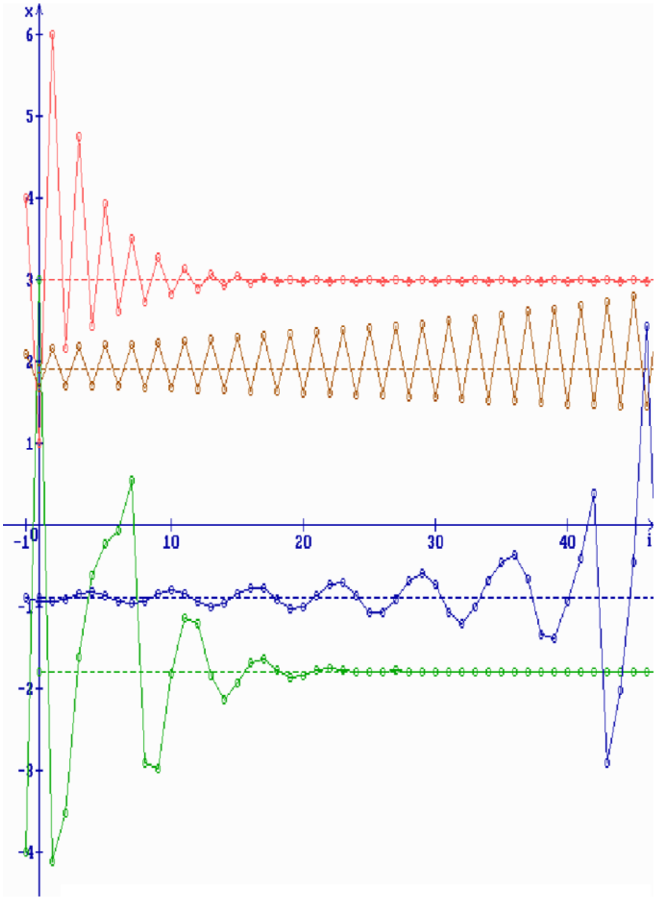


Fig. 7.12 Stable equilibrium points $\hat{x} = 3$ and $\hat{x} = -1.8$, unstable $\hat{x} = 1.93$ and $\hat{x} = -0.9$

Example 7.7 Consider the equation

$$x_{i+1} = \frac{\mu + ax_{i-1}}{\lambda + x_i} + \sigma(x_i - \hat{x})\xi_{i+1} \tag{7.117}$$

(its particular cases were considered in [1, 64, 85, 259]). Equation (7.117) is a particular case of (7.81) with $k = 1$, $a_0 = b_1 = 0$, $a_1 = a$, $b_0 = 1$. From (7.87)–(7.89) it follows that by the condition $\mu > -\frac{1}{4}(a - \lambda)^2$ it has two equilibrium points

$$\hat{x}_1 = \frac{a - \lambda + S}{2}, \qquad \hat{x}_2 = \frac{a - \lambda - S}{2}, \qquad S = \sqrt{(a - \lambda)^2 + 4\mu}.$$

For the equilibrium point $\hat{x}$ the sufficient conditions (7.105) and (7.106), (7.107) give

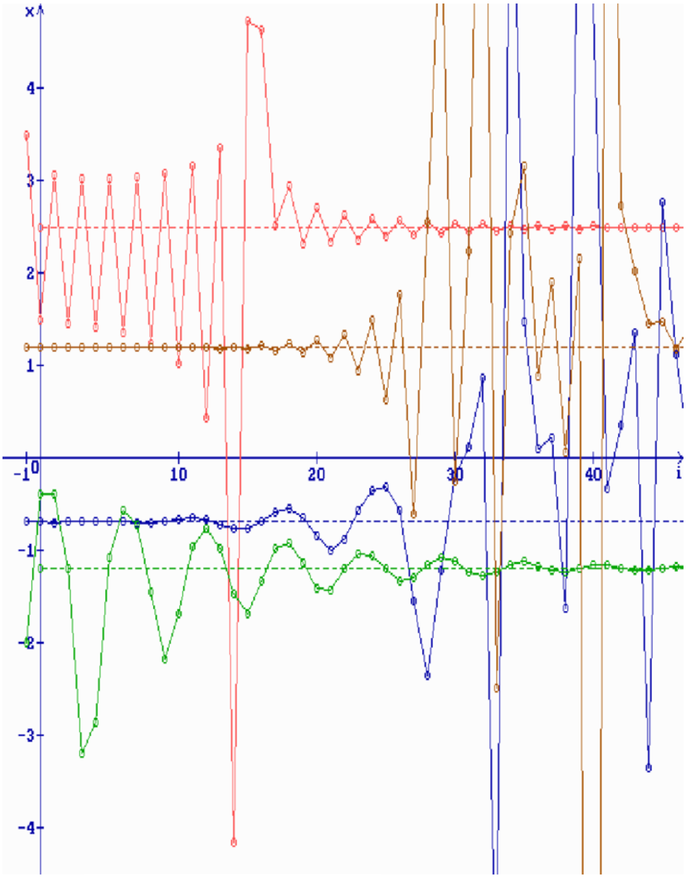


Fig. 7.13 Stable equilibrium point $\hat{x} = 2.5$ and $\hat{x} = -1.2$, unstable $\hat{x} = 1.2$ and $\hat{x} = -0.7$

$$|\hat{x}| + |a| < |\lambda + \hat{x}|\sqrt{1 - \sigma^2}, \tag{7.118}$$

$$2|a| < |\lambda + a| - \sigma^2 \frac{(\lambda + \hat{x})^2}{|\lambda + 2\hat{x} - a|}, \qquad |a - \hat{x}| < |\lambda + \hat{x}|. \tag{7.119}$$

From (7.109) and (7.110) it follows that an equilibrium point $\hat{x}$ of (7.117) is stable in probability if and only if

$$\begin{aligned} &|\lambda + \hat{x}| > |a|, \quad |\hat{x}| < (\lambda + \hat{x} - a)\,\mathrm{sign}(\lambda + \hat{x}), \\ &\sigma^2 < \frac{(\lambda + \hat{x} + a)(\lambda - a)(\lambda + 2\hat{x} - a)}{(\lambda + \hat{x} - a)(\lambda + \hat{x})^2}. \end{aligned} \tag{7.120}$$

For example, for $\hat{x} = \hat{x}_1$ from (7.118) and (7.119) we obtain

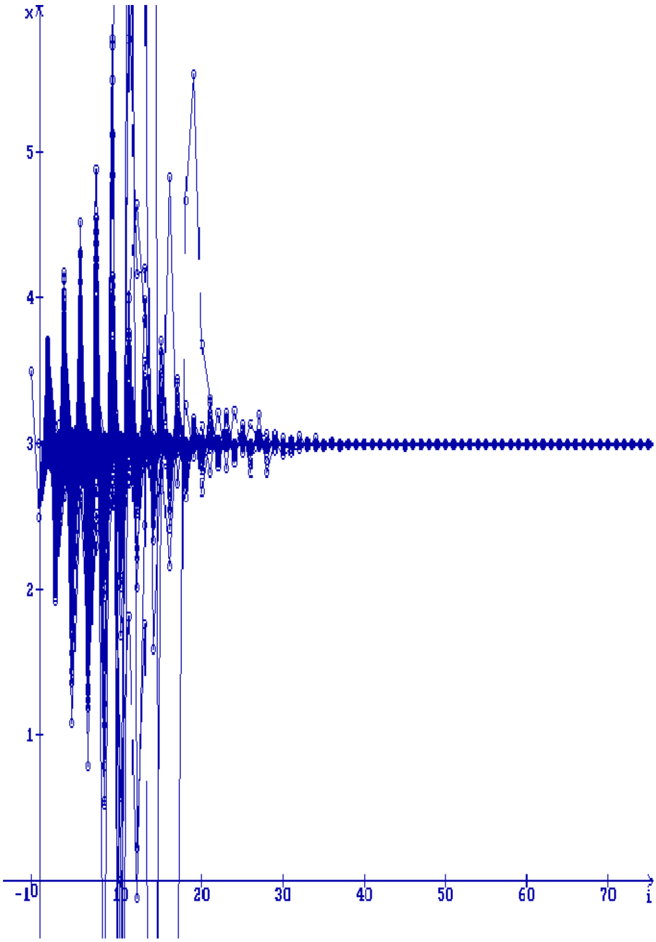


Fig. 7.14 Unstable equilibrium point $\hat{x} = 3$ for $p = 4$, $q = -1$, $\sigma = 0.5$

$$|a - \lambda + S| + 2|a| < |a + \lambda + S|\sqrt{1 - \sigma^2}, \tag{7.121}$$

$$2|a| < \lambda + a - \sigma^2 \frac{(\lambda + a + S)^2}{4S}, \quad \lambda + a > 0. \tag{7.122}$$

From (7.120) it follows that

$$\begin{aligned} &|a + \lambda + S| > 2|a|, \\ &|a - \lambda + S| < (\lambda - a + S)\,\mathrm{sign}(a + \lambda + S), \\ &\sigma^2 < \frac{4S(\lambda - a)(\lambda + 3a + S)}{(\lambda - a + S)(\lambda + a + S)^2}. \end{aligned} \tag{7.123}$$

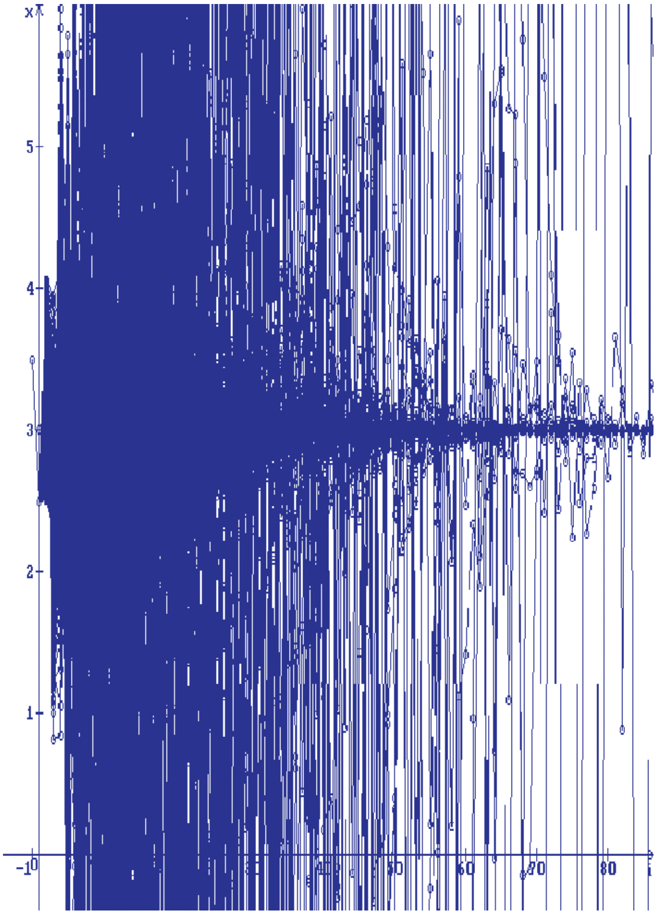


Fig. 7.15 Stable equilibrium point $\hat{x} = 3$ for $p = 4$, $q = -1$, $\sigma = 0.9$

Similarly for $\hat{x} = \hat{x}_2$ from (7.118) and (7.119) we obtain

$$|a - \lambda - S| + 2|a| < |a + \lambda - S|\sqrt{1 - \sigma^2}, \tag{7.124}$$

$$2|a| < |\lambda + a| - \sigma^2 \frac{(\lambda + a - S)^2}{4S}, \quad \lambda + a < 0. \tag{7.125}$$

From (7.120) it follows that

$$\begin{aligned} &|a + \lambda - S| > 2|a|, \\ &|a - \lambda - S| < (\lambda - a - S)\,\mathrm{sign}(a + \lambda - S), \\ &\sigma^2 < \frac{4S(a - \lambda)(\lambda + 3a - S)}{(\lambda - a - S)(\lambda + a - S)^2}. \end{aligned} \tag{7.126}$$

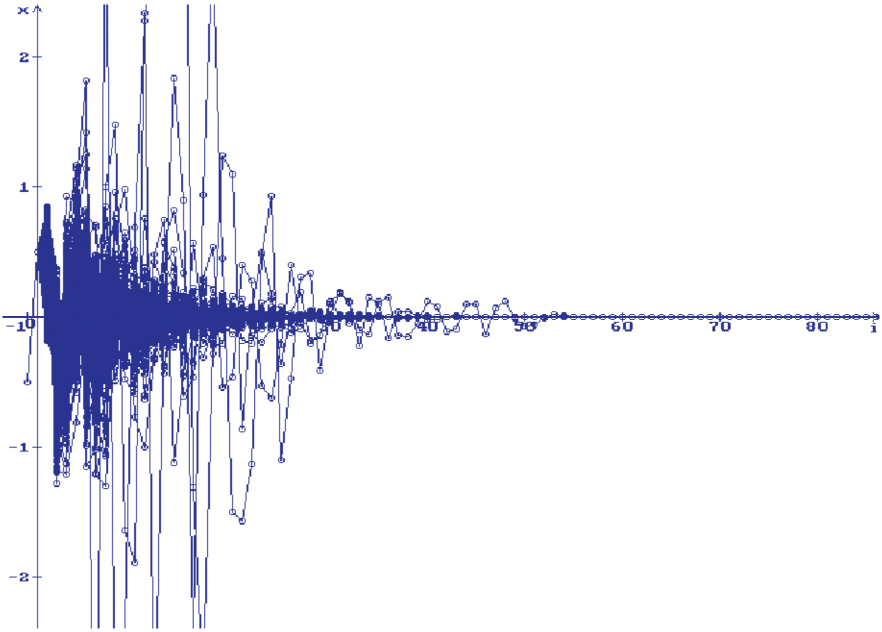


Fig. 7.16 Unstable equilibrium point $\hat{x}_2 = 0$ for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.6$

Put, for example, $\mu = 0$. Then (7.117) has two equilibrium points: $\hat{x}_1 = a - \lambda$, $\hat{x}_2 = 0$. From (7.118)–(7.120) it follows that the equilibrium point $\hat{x}_1$ is unstable and the equilibrium point $\hat{x}_2$ is stable in probability if and only if

$$|\lambda| > \frac{|a|}{\sqrt{1-\sigma^2}}. \tag{7.127}$$

Note that for the particular case of (7.117) by $\mu = 0$, $a = 1$, $\lambda > 0$, $\sigma = 0$ in [259] it is shown that the equilibrium point $\hat{x}_2$ is locally asymptotically stable if $\lambda > 1$ and for the particular case of (7.117) by $\mu = 0$, $a = -\alpha < 0$, $\lambda > 0$, $\sigma = 0$ in [64] it is shown that the equilibrium point $\hat{x}_2$ is locally asymptotically stable if $\lambda > \alpha$. It is easy to see that both these conditions are particular cases of the condition (7.127).

Note that similar results can be obtained for the equation $x_{i+1} = \frac{\mu - a x_i}{\lambda + x_{i-1}}$, which was considered in [1].

In Fig. 7.16 1000 trajectories of (7.117) are shown for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.6$, $x_{-1} = -0.5$, $x_0 = 0.5$. In this case the stability condition (7.127) holds ($2 > 1.25$) and therefore the equilibrium point $\hat{x} = 0$ is stable: all trajectories go to zero. Putting $\sigma = 0.9$, we find that the stability condition (7.127) does not hold ($2 < 2.29$). Therefore, the equilibrium point $\hat{x} = 0$ is unstable: in Fig. 7.17 one can see that 1000 trajectories by the initial condition $x_{-1} = -0.1$, $x_0 = 0.1$ fill the whole space.

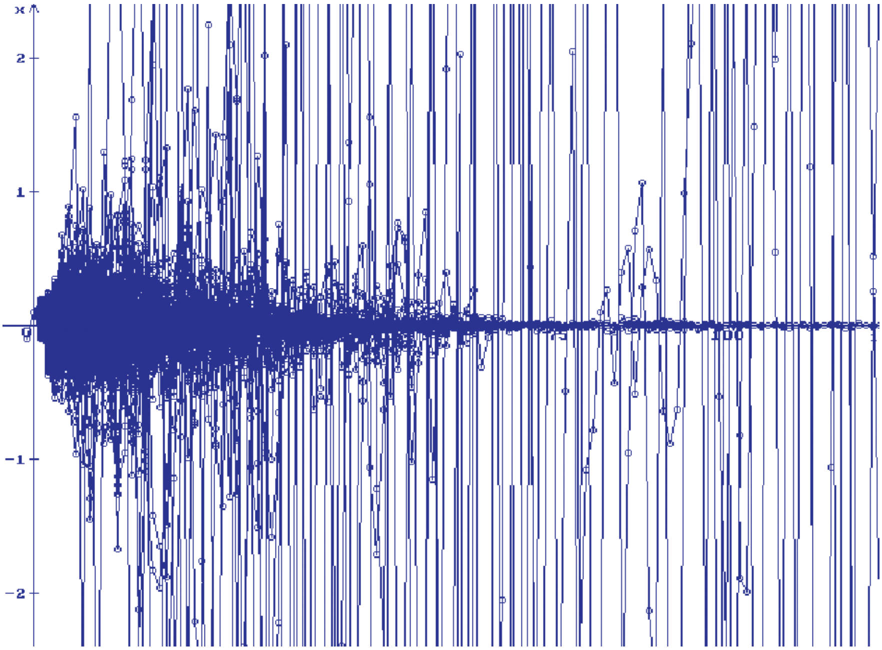


Fig. 7.17 Stability regions, $\sigma = 0$, $\hat{x}_2 = 0$ for $\mu = 0$, $\lambda = -2$, $a = 1$, $\sigma = 0.9$

Example 7.8 Consider the equation

$$x_{i+1} = \frac{p + x_{i-1}}{q x_i + x_{i-1}} + \sigma(x_i - \hat{x})\xi_{i+1} \tag{7.128}$$

which is a particular case of (7.108) with $\mu = p$, $\lambda = 0$, $a_0 = 0$, $a_1 = 1$, $b_0 = q$, $b_1 = 1$. As follows from (7.84) and (7.87)–(7.89) by the conditions $p(q+1) > -\frac{1}{4}$, $q \neq -1$, (7.128) has two equilibrium points

$$\hat{x}_1 = \frac{1+S}{2(q+1)}, \qquad \hat{x}_2 = \frac{1-S}{2(q+1)}, \qquad S = \sqrt{1 + 4p(q+1)}. \tag{7.129}$$

From (7.109) and (7.110) it follows that an equilibrium point $\hat{x}$ of (7.128) is stable in probability if and only if

$$\begin{aligned} &|1 - \hat{x}| < \big|(q+1)\hat{x}\big|, \quad |q\hat{x}| < \big((2+q)\hat{x} - 1\big)\operatorname{sign}\big((q+1)\hat{x}\big), \\ &\sigma^2 < \frac{(1 + q\hat{x})(2\hat{x} - 1)(2(q+1)\hat{x} - 1)}{((2+q)\hat{x} - 1)(q+1)^2\hat{x}^2}. \end{aligned} \tag{7.130}$$

Substituting (7.129) into (7.130), we obtain the stability conditions immediately in terms of the parameters of the equation considered, (7.128): the equilibrium point

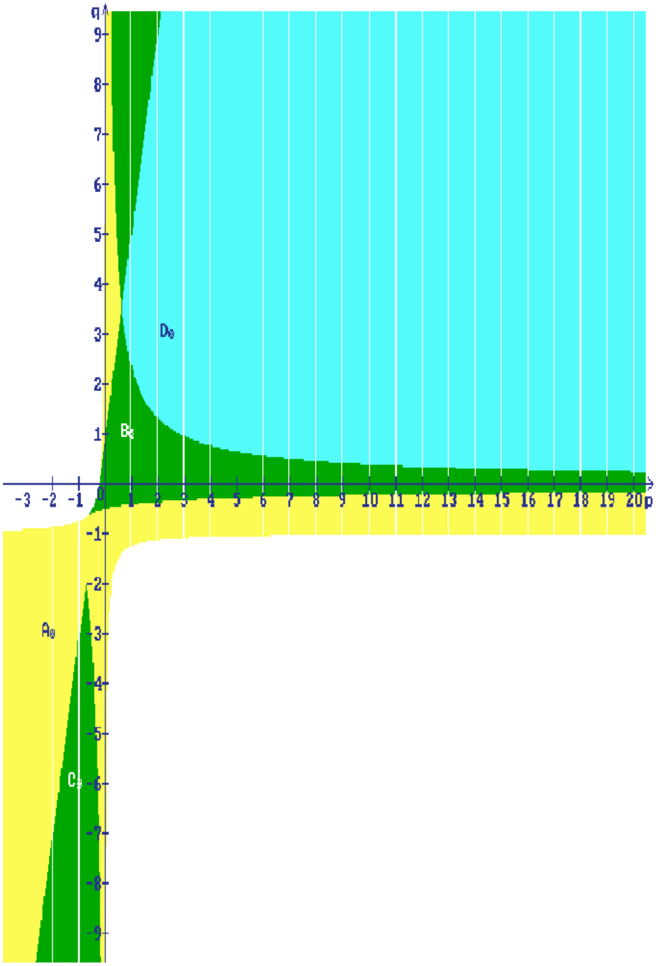


Fig. 7.18 Stability regions, $\sigma = 0$

$\hat{x}_1$ is stable in probability if and only if

$$p \in \begin{cases} (\frac{q-1}{4}, \infty), & q \geq 0, \\ (-\frac{1}{4(q+1)}, \frac{2}{q} + \frac{1}{q^2}), & q \in (-\frac{2}{3}, 0), \end{cases} \Bigg| \quad \sigma^2 < \frac{4S(S-q)((S+3)q+2)}{(S+1)^2(q+1)((q+2)S-q)}, \tag{7.131}$$

the equilibrium point $\hat{x}_2$ is stable in probability if and only if

$$p \in \begin{cases} (\frac{2}{q} + \frac{1}{q^2}, \infty), & q > 0, \\ (\frac{q-1}{4}, \frac{2}{q} + \frac{1}{q^2}), & q < -2, \end{cases} \Bigg| \quad \sigma^2 < \frac{4S(S+q)((S-3)q-2)}{(S-1)^2(q+1)((q+2)S+q)}. \tag{7.132}$$

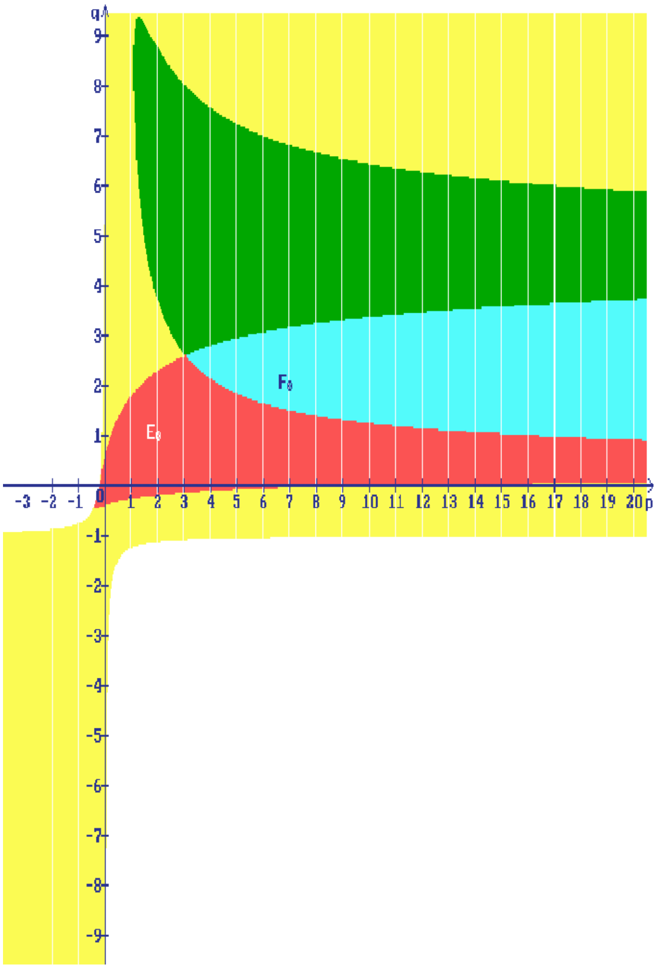


Fig. 7.19 Stability regions, $\sigma = 0.7$

Note that in [147] (7.128) was considered with $\sigma = 0$ and positive p, q. There it was shown that the equilibrium point $\hat{x}_1$ is locally asymptotically stable if and only if $q < 4p + 1$ which is part of the conditions (7.131).

In Fig. 7.18 the region where the points of equilibrium are absent (white region), the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are though being unstable (yellow region), the region where the point of equilibrium $\hat{x}_1$ is stable only (red region), the region where the point of equilibrium $\hat{x}_2$ is stable only (green region) and the region where both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ are stable (cyan region) are shown in the space of (p, q). All regions are obtained via the conditions (7.131) and (7.132) for $\sigma = 0$. In Fig. 7.19 similar regions are shown for $\sigma = 0.7$.

Consider the point A (Fig. 7.18) with $p = -2$, $q = -3$. In this point both equilibrium points $\hat{x}_1 = -1.281$ and $\hat{x}_2 = 0.781$ are unstable. In Fig. 7.20 two trajecto-

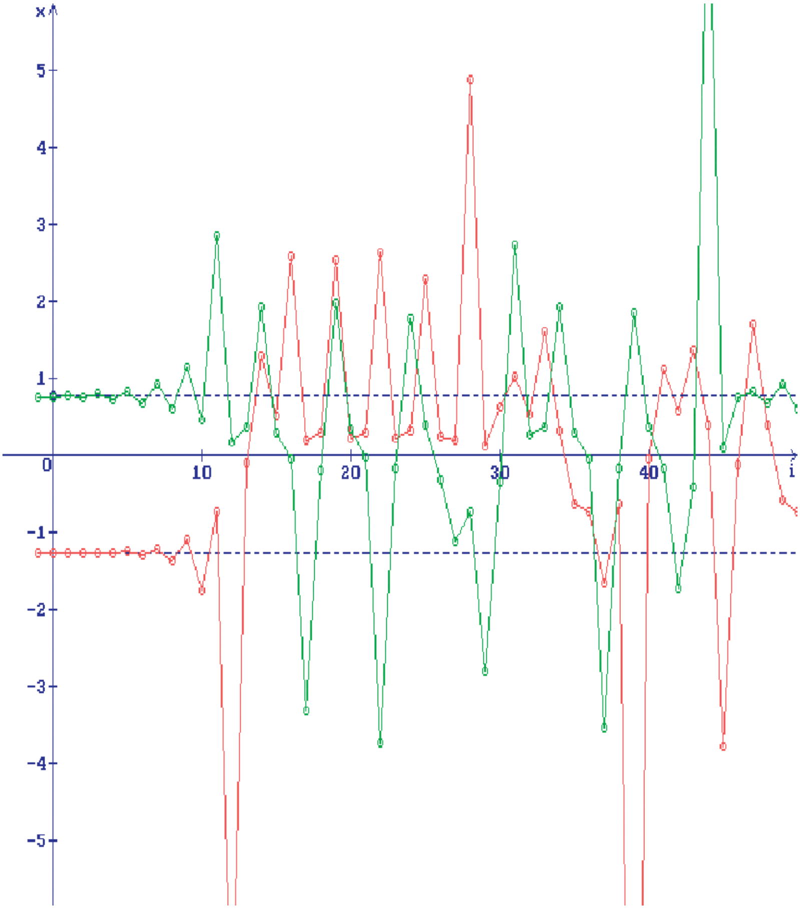


Fig. 7.20 Unstable equilibrium points $\hat{x}_1 = -1.281$ and $\hat{x}_2 = 0.781$ for $p = -2$, $q = -3$

ries of the solutions of (7.128) are shown with the initial conditions $x_{-1} = -1.28$, $x_0 = -1.281$ and $x_{-1} = 0.771$, $x_0 = 0.77$. In Fig. 7.21 two trajectories of solutions of (7.128) with the initial conditions $x_{-1} = 4$, $x_0 = -3$ and $x_{-1} = -0.51$, $x_0 = -0.5$ are shown in point B (Fig. 7.18) with $p = q = 1$. One can see that the equilibrium point $\hat{x}_1 = 1$ is stable and the equilibrium point $\hat{x}_2 = -0.5$ is unstable. In point C (Fig. 7.18) with $p = -1$, $q = -6$ the equilibrium point $\hat{x}_1 = -0.558$ is unstable and the equilibrium point $\hat{x}_2 = 0.358$ is stable. Two corresponding trajectories of solutions are shown in Fig. 7.22 with the initial conditions $x_{-1} = x_0 = -0.55$ and $x_{-1} = -4$, $x_0 = 5$. In point D (Fig. 7.18) with $p = 2.5$, $q = 3$ both equilibrium points $\hat{x}_1 = 0.925$ and $\hat{x}_2 = -0.675$ are stable. Two corresponding trajectories of solutions are shown in Fig. 7.23 with the initial conditions $x_{-1} = 2.1$, $x_0 = 0.2$ and $x_{-1} = -0.2$, $x_0 = -1.4$.

Consider the behavior of the equilibrium points of (7.128) by stochastic perturbations with $\sigma = 0.7$. In Fig. 7.24 the trajectories of solutions are shown for

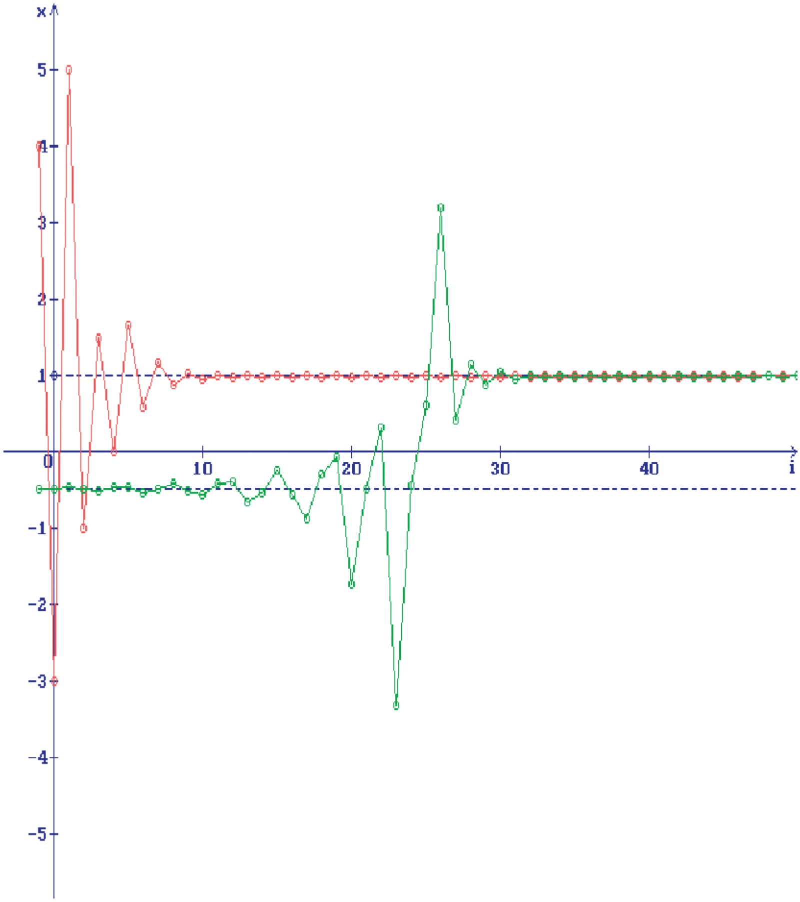


Fig. 7.21 Stable equilibrium point $\hat{x}_1 = 1$ and unstable $\hat{x}_2 = -0.5$ for $p = 1$, $q = 1$

$p = 2$, $q = 1$ (point E in Fig. 7.19) with the initial conditions $x_{-1} = 1.5$, $x_0 = 1$ and $x_{-1} = x_0 = -0.78$. One can see that the equilibrium point $\hat{x}_1 = 1.281$ (red trajectories) is stable and the equilibrium point $\hat{x}_2 = -0.781$ (green trajectories) is unstable. In Fig. 7.25 the trajectories of solutions are shown for $p = 7$, $q = 2$ (point F in Fig. 7.19) with the initial conditions $x_{-1} = 1.5$, $x_0 = 1.9$ and $x_{-1} = -1.4$, $x_0 = -1.3$. In this case both equilibrium points $\hat{x}_1 = 1.703$ (red trajectories) and $\hat{x}_2 = -1.37$ (green trajectories) are stable.

7.4 Almost Sure Stability

In this section almost sure asymptotic stability of the trivial solution of a nonlinear scalar stochastic difference equation is studied. Sufficient criteria for stability are

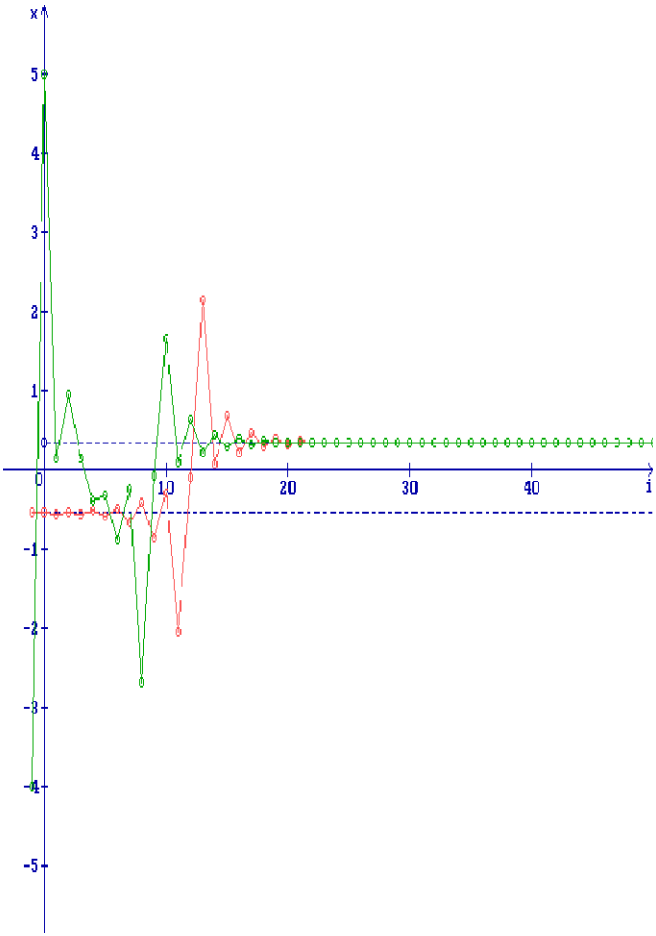


Fig. 7.22 Unstable equilibrium point $\hat{x}_1 = -0.558$ and stable $\hat{x}_2 = 0.358$ for $p = -1$, $q = -6$

obtained by virtue of the procedure of the construction of the Lyapunov functionals, martingale decomposition and semi-martingale convergence theorems.

7.4.1 Auxiliary Statements and Definitions

Consider the stochastic difference equation

$$\begin{aligned} x_{i+1} &= x_i + \kappa_i \Phi(x_i) - a_i \Phi(x_{i+1}) + f_i(x_i) \\ &\quad + g_i(x_i, x_{i-1}, \ldots) + \sigma_i(x_i, x_{i-1}, \ldots)\xi_{i+1}, \\ &i \in Z, \; x_j = \varphi_j, \; j \in Z_0. \end{aligned} \tag{7.133}$$

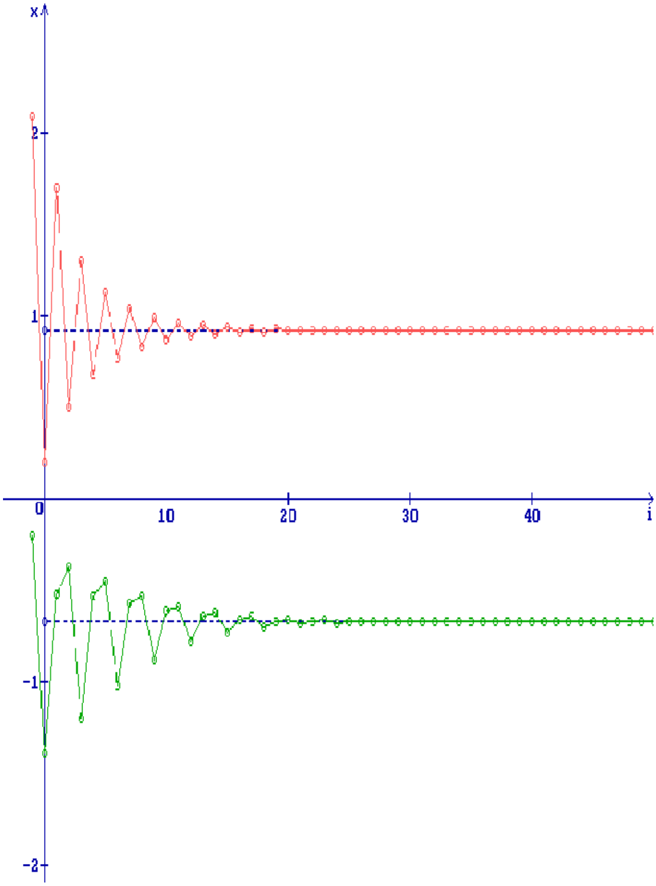


Fig. 7.23 Stable equilibrium points $\hat{x}_1 = 0.925$ and $\hat{x}_2 = -0.675$ for $p = 2.5$, $q = 3$

The concept of almost sure asymptotic stability for the solution of this equation is defined as follows.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic complete probability space, $\mathfrak{F}_i \in \mathfrak{F}$, $i \in Z$, a family of σ-algebras, ξ_i, $i \in Z$, martingale-differences [253], non-random parameters κ_i, a_i and the functions $\Phi(x)$, $f_i(x)$ be defined for $x \in \mathbf{R}^1$, $i \in Z$; let the functionals $g_i(x_0, x_1, \ldots)$ and $\sigma_i(x_0, x_1, \ldots)$ be defined on $Z \times S$, where S is a space of sequences $\{x_0, x_1, \ldots\}$, $\Phi(0) = 0$, $f_i(0) = 0$, $g_i(0, 0, \ldots) = 0$, $\sigma_i(0, 0, \ldots) = 0$. The abbreviation "a.s." is used for such wordings as "**P**-almost sure" or "**P**-almost surely", respectively, wherever convenient.

Definition 7.2 The trivial solution of (7.133) is called globally a.s. asymptotically stable if for each nontrivial (a.s.) initial function, which is independent on the σ-algebra $\sigma(\xi_i : i \in Z)$, the solution x_i satisfies the condition $\mathbf{P}(\lim_{i\to\infty} x_i = 0) = 1$.

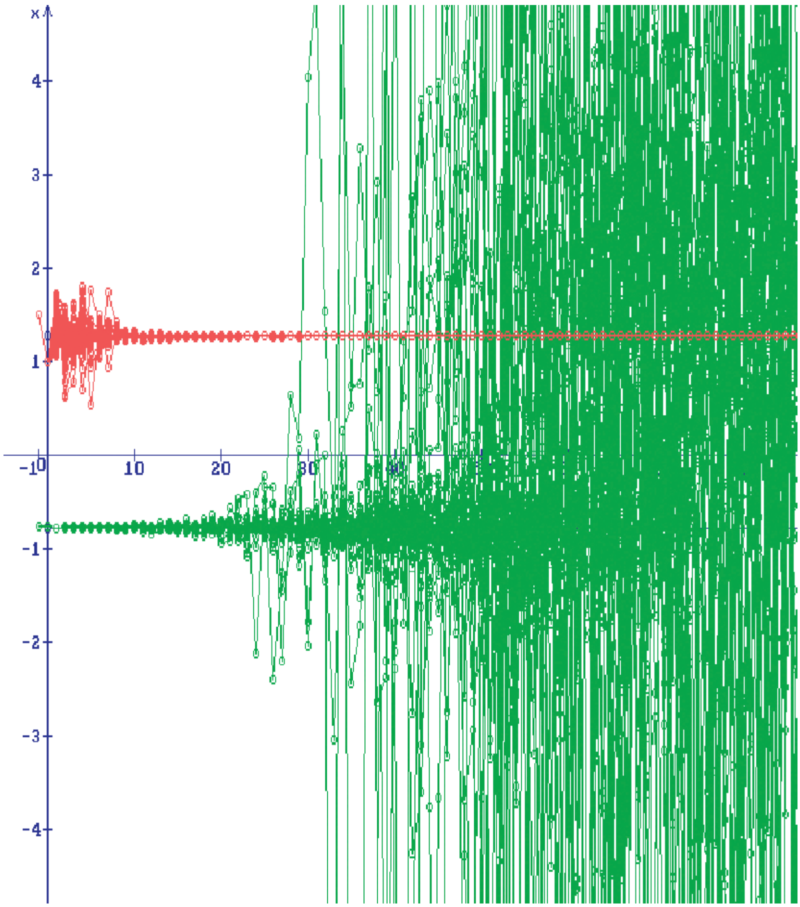


Fig. 7.24 Stable equilibrium point $\hat{x}_1 = 1.281$ and unstable $\hat{x}_2 = -0.781$ for $p = 2$, $q = 1, \sigma = 0.7$

Let $\mathfrak{B}(\mathbf{R}^1)$ be the set of all Borel-sets of the set $\mathbf{R}^1$. The following lemma is a generalization of Doob decomposition of sub-martingales (for details, see [174]).

Lemma 7.2 *Let the sequence ξ_i, $i \in Z$, be a $\mathfrak{F}_i$-martingale-difference. Then there is a $\mathfrak{F}_i$-martingale-difference μ_i, $i \in Z$, and a positive $(\mathfrak{F}_{i-1}, \mathfrak{B}(\mathbf{R}^1))$-measurable (i.e. predictable) stochastic sequence η_i, $i \in Z$, such that, for every $i \in Z$, almost surely $\xi_i^2 = \mu_i + \eta_i$ and the process η_i, $i \in Z$, can be represented in the form $\eta_i = \mathbf{E}[\xi_i^2 \mid \mathfrak{F}_{i-1}]$. Moreover, if ξ_i, $i \in Z$, are independent random variables, then η_i, $i \in Z$, is a non-random sequence such that*

$$\eta_i = \mathbf{E}\xi_i^2, \qquad \mu_i = \xi_i^2 - \mathbf{E}\xi_i^2. \tag{7.134}$$

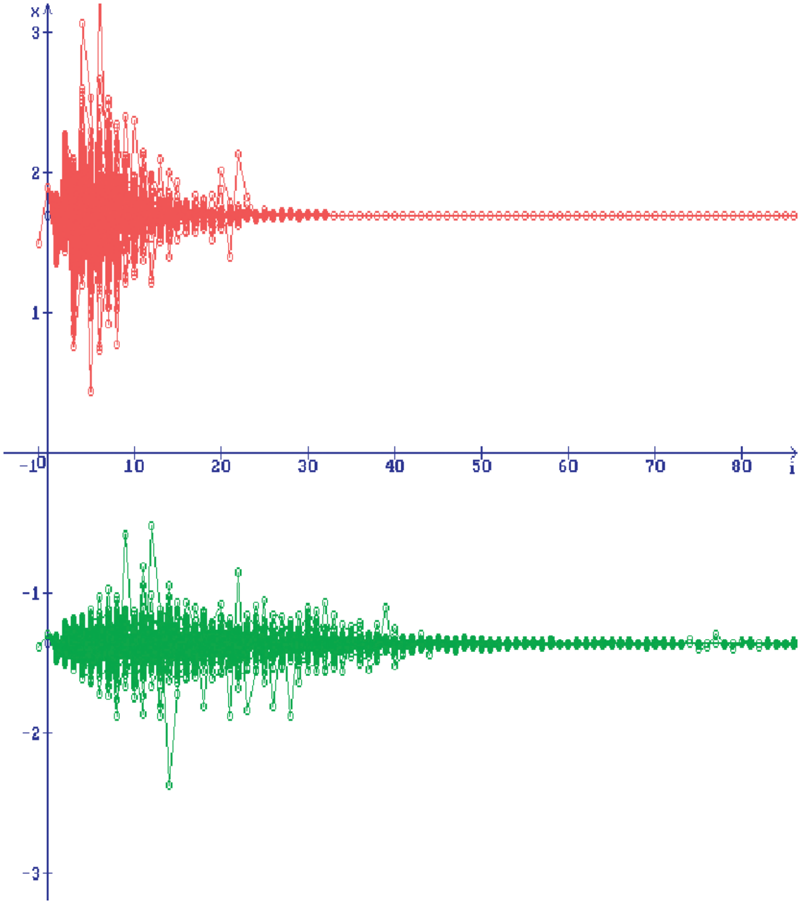


Fig. 7.25 Stable equilibrium points $\hat{x}_1 = 1.703$ and $\hat{x}_2 = -1.37$ for $p = 7$, $q = 2$, $\sigma = 0.7$

Lemma 7.3 *Let W_i, $i \in Z$, be a nonnegative $(\mathfrak{F}_i, \mathfrak{B}(\mathbf{R}^1))$-measurable process with the properties $\mathbf{E}W_i < \infty$, $i \in Z$, and*

$$W_{i+1} \le W_i + U_i - V_i + \zeta_{i+1}, \quad i \in Z, \tag{7.135}$$

where ζ_i is an $\mathfrak{F}_i$-martingale-difference, U_i and V_i are nonnegative $(\mathfrak{F}_i, \mathfrak{B}(\mathbf{R}^1))$-measurable processes with $\mathbf{E}U_i < \infty$, $\mathbf{E}V_i < \infty$, $i \in Z$. Then

$$\left\{\omega \in \Omega : \sum_{i=0}^{\infty} U_i < \infty\right\} \subseteq \left\{\omega \in \Omega : \sum_{i=0}^{\infty} V_i < \infty\right\} \cap \left\{\omega \in \Omega : W_i(\omega) \to\right\}.$$

Here $\{\omega \in \Omega : W_i(\omega) \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim_{i\to\infty} W_i(\omega)$ exists and is finite.

Remark 7.14 Lemma 7.3 remains correct if $V_i \geq 0$ only for $i \geq i_0$, where $i_0 \in Z$ is non-random.

Lemma 7.4 *The increments $\Delta V_i = V_{i+1} - V_i$ of the functional*

$$V_i = \sum_{j=1}^{\infty} \alpha_{j,i-1} f(x_{i-j}), \quad \alpha_{ji} = \sum_{k=j}^{\infty} \beta_{ki},$$

can be represented by

$$\Delta V_i = \alpha_{1i} f(x_i) + \sum_{j=1}^{\infty} \left(\sum_{l=j+1}^{\infty} (\beta_{li} - \beta_{l,i-1}) - \beta_{j,i-1} \right) f(x_{i-j}).$$

If in addition β_{ji} is non-increasing in i for all $j \in Z$, and $f(x)$ is a nonnegative function, then

$$\Delta V_i \leq \alpha_{1i} f(x_i) - \sum_{j=1}^{\infty} \beta_{j,i-1} f(x_{i-j}), \quad i \in Z.$$

Proof It is enough to note that

$$\begin{aligned} \Delta V_i &= \sum_{j=1}^{\infty} \alpha_{ji} f(x_{i+1-j}) - \sum_{j=1}^{\infty} \alpha_{j,i-1} f(x_{i-j}) \\ &= \sum_{j=0}^{\infty} \alpha_{j+1,i} f(x_{i-j}) - \sum_{j=1}^{\infty} \alpha_{j,i-1} f(x_{i-j}) \\ &= \alpha_{1i} f(x_i) + \sum_{j=1}^{\infty} (\alpha_{j+1,i} - \alpha_{j,i-1}) f(x_{i-j}) \\ &= \alpha_{1i} f(x_i) + \sum_{j=1}^{\infty} \left(\sum_{l=j+1}^{\infty} (\beta_{li} - \beta_{l,i-1}) - \beta_{j,i-1} \right) f(x_{i-j}). \end{aligned}$$

□

7.4.2 Stability Theorems

Below it is supposed that the parameters of (7.133) satisfy the following major hypotheses.

(H0) a_i is a non-increasing sequence of non-random, nonnegative real numbers and κ_i is a sequence of non-random real numbers satisfying the condition

$$a_i > |\kappa_i|, \quad i \in Z. \tag{7.136}$$

(H1) The function $\Phi \in \{\mathbf{R}^1 \to \mathbf{R}^1\}$ is continuous and satisfies the following conditions: if $\Phi(x) = 0$ then $x = 0$ and

$$x\Phi(x) \geq cx^{2m}, \tag{7.137}$$

where m is a positive integer and $c > 0$.

(H2) There are nonnegative non-random numbers δ_i, ν_i, $\gamma_{ji}^{(0)}$, $\gamma_{ji}^{(1)}$, $\gamma_{ji}^{(2)}$ such that

$$f_i^2(x_i) \leq \delta_i \Phi^2(x_i), \qquad \left|g_i(x_i, x_{i-1}, \ldots)\right| \leq \sum_{j=0}^{\infty} \gamma_{ji}^{(0)} |x_{i-j}|^{2m-1}, \tag{7.138}$$

$$\sigma_i^2(x_i, x_{i-1}, \ldots) \leq \nu_i + \sum_{j=0}^{\infty} \gamma_{ji}^{(1)} x_{i-j} \Phi(x_{i-j}) + \sum_{j=0}^{\infty} \gamma_{ji}^{(2)} \Phi^2(x_{i-j}), \tag{7.139}$$

$$\sum_{i=0}^{\infty} \nu_i \eta_{i+1} < \infty, \quad \text{a.s.} \tag{7.140}$$

(H3)

$$\varepsilon_i^{(1)} = 2\left(a_i - \kappa_i - \sqrt{\delta_i} - \frac{\rho_i^{(0)}}{c}\right) - \eta_{i+1}\rho_i^{(1)} \geq 0, \tag{7.141}$$

$$\begin{aligned} \varepsilon_i^{(2)} &= a_i^2 - \kappa_i^2 - \eta_{i+1}\rho_i^{(2)} \\ &\quad - |\kappa_i|\left[2\sqrt{\delta_i} + \left(1 + \frac{1}{c^2}\right)\rho_i^{(0)}\right] - \left(\sqrt{\delta_i} + \frac{\rho_i^{(0)}}{c}\right)^2 \geq 0, \end{aligned} \tag{7.142}$$

where

$$\rho_i^{(l)} = \sum_{j=0}^{\infty} \gamma_{ji}^{(l)}, \quad l = 0, 1, 2, \ i \in Z. \tag{7.143}$$

(H4) There exist $\varepsilon > 0$ and $i_0 \in Z$ such that $\varepsilon_i^{(1)} + \varepsilon_i^{(2)} > \varepsilon > 0$ for all $i \geq i_0$.

(H5) All $\beta_{ji}^{(1)}$ and $\beta_{ji}^{(2)}$ are non-increasing in i, where

$$\begin{aligned} \beta_{ji}^{(1)} &= \gamma_{ji}^{(1)} \eta_{i+1} + \gamma_{ji}^{(0)} \frac{2m-1}{mc}, \\ \beta_{ji}^{(2)} &= \gamma_{ji}^{(2)} \eta_{i+1} + \gamma_{ji}^{(0)} \left(|\kappa_i| + \rho_i^{(0)} + c\sqrt{\delta_i}\right) \frac{1}{c^2}. \end{aligned} \tag{7.144}$$

Despite of its apparent (visible) complexity, hypotheses (H0)–(H4) are fulfilled in many cases. We now present three examples satisfying those major hypotheses.

Example 7.9 Let $a_i \equiv a > 0$, $\kappa_i \equiv \kappa > 0$, $a > \kappa$, $f_i \equiv 0$ and therefore $\delta_i \equiv 0$, $\Phi(x) = x^{2m-1}$ and therefore $c = 1$, $g_i \equiv 0$, $\gamma_{ji}^{(l)} \equiv 0$, $l = 0, 1, 2$, $\eta_i \equiv 1$, $\nu_i = \frac{1}{(i+1)^2}$, $j, i \in Z$.

In this case (7.133) takes the form

$$x_{i+1} = x_i + \kappa_i x_i^{2m-1} - a_i x_{i+1}^{2m-1} + \frac{1}{i+1}\xi_{i+1}, \quad i \in Z,$$

and

$$\varepsilon_i^{(1)} = 2(a-\kappa) > 0, \qquad \varepsilon_i^{(2)} = a^2 - \kappa^2 > 0,$$
$$\varepsilon = \min\{\varepsilon_i^{(1)}\varepsilon_i^{(2)}\}, \qquad \beta_{ji}^{(1)} = \beta_{ji}^{(2)} \equiv 0.$$

Example 7.10 Let $a_i \equiv a > 0$, $\kappa_i \equiv \kappa > 0$, $a > \kappa$, $\Phi(x) = x^{2m-1}$ and therefore $c = 1$, $\delta_i \equiv \delta = (\frac{a-\kappa}{2})^2$, $f_i(x) = f(x)x^{2m-1}$, where $f^2(x) \le \delta$ for all $x \in \mathbf{R}^1$, $\eta_i \equiv 1$, $\nu_i = \frac{1}{(i+1)^2}$, $\gamma_{ji}^{(l)} = \frac{\gamma(l)}{(j+1)^2}$, $l = 0, 1, 2$, $j, i \in Z$.

Put

$$S = \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} = 1.645$$

and suppose that

$$\gamma(0) < \frac{a-\kappa}{4S}, \qquad \gamma(1) < \frac{a-\kappa}{2S}, \qquad \gamma(2) < \frac{(a-\kappa)(7a+\kappa)}{16S}. \tag{7.145}$$

From (7.143) and (7.145) it follows that $\rho_i^{(l)} = \gamma(l)S$, $l = 0, 1, 2$, and $(2\gamma(0) + \gamma(1))S < a - \kappa$. So

$$\varepsilon_i^{(1)} \equiv \varepsilon^{(1)} = 2\big(a - \kappa - \sqrt{\delta} - \gamma(0)S\big) - \gamma(1)S = a - \kappa - \big(2\gamma(0) + \gamma(1)\big)S > 0.$$

An estimation of $\varepsilon_i^{(2)}$ yields

$$\begin{aligned}
\varepsilon_i^{(2)} \equiv \varepsilon^{(2)} &= a^2 - \kappa^2 - \gamma(2)S - |\kappa|\big(2\sqrt{\delta} + 2\gamma(0)S\big) - \big(\sqrt{\delta} + \gamma(0)S\big)^2 \\
&= a^2 - \kappa^2 - \kappa(a-\kappa) - \gamma(2)S - 2\kappa\gamma(0)S - \left(\frac{a-\kappa}{2} + \gamma(0)S\right)^2 \\
&> (a-\kappa)a - \frac{(a-\kappa)(7a+\kappa)}{16} - \kappa\frac{a-\kappa}{2} - \left(\frac{a-\kappa}{2} + \frac{a-\kappa}{4}\right)^2 \\
&= (a-\kappa)\left(a - \frac{7a+\kappa}{16} - \frac{\kappa}{2} - \frac{9(a-\kappa)}{16}\right) = 0.
\end{aligned}$$

Thus, $\varepsilon = \min\{\varepsilon^{(1)}, \varepsilon^{(2)}\}$ and $\beta_{ji}^{(l)}$, $l = 1, 2$, does not depend on i.

Example 7.11 Let $a_i = a + \kappa_i$, $a > 0$, $\kappa_i \ge 0$, $i \in Z$, let κ_i be decreasing, $\Phi(x) = \psi(x)x^{2m-1}$, $\psi(x) > 1$, be an arbitrary continuous function and therefore $c = 1$, $f_i(x) = \psi_i(x)x^{2m-1}$, where $|\psi_i(x)|^2 \le \delta_i < \frac{a^2}{4}$ for all $x \in \mathbf{R}^1$, $i \in Z$, $\eta_i = \frac{1}{i+1}$, $\nu_i = \frac{1}{i+1}$, $\gamma_{ji}^{(l)} = \frac{\gamma(l)}{(i+1)(j+1)^2}$, $\gamma(l) > 0$, $l = 0, 1, 2$, $j, i \in Z$.

We have

$$\rho_i^{(l)} = \frac{1}{i+1}\gamma(l)S, \quad l = 0, 1, 2.$$

So, if $i \to \infty$ then

$$\varepsilon_i^{(1)} > 2\left(a - \frac{a}{2} - \frac{S}{i+1}\gamma(0)\right) - \frac{S\gamma(1)}{(i+2)(i+1)} = a + O\left(i^{-1}\right) \to a.$$

For the estimation of $\varepsilon_i^{(2)}$ we obtain

$$\begin{aligned}\varepsilon_i^{(2)} &> (a+\kappa_i)^2 - \kappa_i^2 - \frac{S}{(i+2)(i+1)}\gamma(2) \\ &\quad - |\kappa_i|\left(a + \frac{2S}{i+1}\gamma(0)\right) - \left(\frac{a}{2} + \frac{S}{i+1}\gamma(0)\right)^2 \\ &= \frac{3}{4}a^2 + a\kappa_i + O\left(i^{-1}\right) \geq \frac{3}{4}a^2 + O\left(i^{-1}\right),\end{aligned}$$

which also tends to a positive limit as $i \to \infty$.

Consequently, we can recognize that the hypotheses (H0)–(H5) are satisfied in several meaningful examples.

Theorem 7.6 *Let ξ_i, $i \in Z$, be square-integrable, independent random variables with $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = \eta_i$ and the hypotheses* (H0)–(H5) *be fulfilled. Then the trivial solution of* (7.133) *is globally a.s. asymptotically stable.*

Proof Rewrite equation (7.133) in the equivalent form

$$\begin{aligned} & y_{i+1} + (a_i - a_{i+1})\Phi(x_{i+1}) \\ &\quad = y_i - (a_i - \kappa_i)\Phi(x_i) \\ &\qquad + f_i(x_i) + g_i(x_i, x_{i-1}, \ldots) + \sigma_i(x_i, x_{i-1}, \ldots)\xi_{i+1}, \end{aligned} \tag{7.146}$$

where

$$y_i = x_i + a_i\Phi(x_i). \tag{7.147}$$

Following the procedure of the construction of the Lyapunov functionals, we will construct the Lyapunov functional V_i for (7.133) in the form $V_i = V_{1i} + V_{2i}$ with

$$V_{1i} = y_i^2 = \left(x_i + a_i\Phi(x_i)\right)^2.$$

Note that via hypothesis (H0) and (7.147) and (7.137)

$$\begin{aligned}&\big(y_{i+1}+(a_i-a_{i+1})\Phi(x_{i+1})\big)^2\\&\quad=y_{i+1}^2+2(a_i-a_{i+1})y_{i+1}\Phi(x_{i+1})+(a_i-a_{i+1})^2\Phi^2(x_{i+1})\\&\quad\ge y_{i+1}^2+2(a_i-a_{i+1})\big(x_{i+1}\Phi(x_{i+1})+a_i\Phi^2(x_{i+1})\big)\ge y_{i+1}^2.\end{aligned}$$

So, applying (7.146), (7.147) and (7.134) and the equality $y_i-(a_i-\kappa_i)\Phi(x_i)=x_i+\kappa_i\Phi(x_i)$ for the estimation of the increments $\Delta V_{1i}=y_{i+1}^2-y_i^2$ we obtain

$$\begin{aligned}\Delta V_{1i}&\le\big(y_{i+1}+(a_i-a_{i+1})\Phi(x_{i+1})\big)^2-y_i^2\\&=\big[y_i-(a_i-\kappa_i)\Phi(x_i)+f_i(x_i)+g_i(x_i,x_{i-1},\dots)\\&\quad+\sigma_i(x_i,x_{i-1},\dots)\xi_{i+1}\big]^2-y_i^2\\&=-2(a_i-\kappa_i)y_i\Phi(x_i)+(a_i-\kappa_i)^2\Phi^2(x_i)\\&\quad+2\big(x_i+\kappa_i\Phi(x_i)\big)\big[f_i(x_i)+g_i(x_i,x_{i-1},\dots)\big]\\&\quad+\big[f_i(x_i)+g_i(x_i,x_{i-1},\dots)\big]^2+\sigma_i^2(x_i,x_{i-1},\dots)\eta_{i+1}+\zeta_{i+1},\end{aligned}\tag{7.148}$$

where

$$\begin{aligned}\zeta_{i+1}&=2\big[x_i+\kappa_i\Phi(x_i)+f_i(x_i)+g_i(x_i,x_{i-1},\dots)\big]\sigma_i(x_i,x_{i-1},\dots)\xi_{i+1}\\&\quad+\sigma_i^2(x_i,x_{i-1},\dots)\mu_{i+1}\end{aligned}$$

and μ_i is the martingale-difference defined by (7.134).

Via (7.147) and Lemma 1.2 we continue the estimation (7.148) with some $\alpha_i>0$

$$\begin{aligned}\Delta V_{1i}&\le-2(a_i-\kappa_i)\big(x_i+a_i\Phi(x_i)\big)\Phi(x_i)+(a_i-\kappa_i)^2\Phi^2(x_i)\\&\quad+2\big(x_i+\kappa_i\Phi(x_i)\big)\big[f_i(x_i)+g_i(x_i,x_{i-1},\dots)\big]+\sigma_i^2(x_i,x_{i-1},\dots)\eta_{i+1}\\&\quad+(1+\alpha_i)f_i^2(x_i)+\big(1+\alpha_i^{-1}\big)g_i^2(x_i,x_{i-1},\dots)+\zeta_{i+1}.\end{aligned}\tag{7.149}$$

Using (7.143) and the inequalities (7.138) and (7.137), we get

$$\begin{aligned}g_i^2(x_i,x_{i-1},\dots)&\le\rho_i^{(0)}\sum_{j=0}^{\infty}\gamma_{ji}^{(0)}|x_{i-j}|^{2(2m-1)}\le\frac{\rho_i^{(0)}}{c^2}\sum_{j=0}^{\infty}\gamma_{ji}^{(0)}\Phi^2(x_{i-j})\\&=\frac{\rho_i^{(0)}}{c^2}\left(\gamma_{0i}^{(0)}\Phi^2(x_i)+\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}\Phi^2(x_{i-j})\right),\end{aligned}\tag{7.150}$$

$$\begin{aligned}\big|f_i(x_i)\big(x_i+\kappa_i\Phi(x_i)\big)\big|&\le\sqrt{\delta_i}\big|\Phi(x_i)\big|\big|x_i+\kappa_i\Phi(x_i)\big|\\&\le\sqrt{\delta_i}\big[x_i\Phi(x_i)+|\kappa_i|\Phi^2(x_i)\big].\end{aligned}\tag{7.151}$$

Via Lemma 1.1 we obtain

$$\begin{aligned}
&\big|g_i(x_i,x_{i-1},\dots)x_i\big| \\
&\quad\le \sum_{j=0}^{\infty}\gamma_{ji}^{(0)}|x_{i-j}|^{2m-1}|x_i| \\
&\quad\le \frac{1}{2m}\sum_{j=0}^{\infty}\gamma_{ji}^{(0)}\big(|x_i|^{2m}+(2m-1)|x_{i-j}|^{2m}\big) \\
&\quad= \frac{1}{2m}\Bigg[\big(\rho_i^{(0)}+(2m-1)\gamma_{0i}^{(0)}\big)|x_i|^{2m}+(2m-1)\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}|x_{i-j}|^{2m}\Bigg] \\
&\quad\le \frac{1}{2mc}\Bigg[\big(\rho_i^{(0)}+(2m-1)\gamma_{0i}^{(0)}\big)x_i\Phi(x_i)+(2m-1)\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}x_{i-j}\Phi(x_{i-j})\Bigg].
\end{aligned} \tag{7.152}$$

Besides, via Lemma 1.1 and (7.137)

$$\begin{aligned}
\big|g_i(x_i,x_{i-1},\dots)\Phi(x_i)\big| &\le \sum_{j=0}^{\infty}\gamma_{ji}^{(0)}|x_{i-j}|^{2m-1}\big|\Phi(x_i)\big| \\
&\le \frac{1}{2}\sum_{j=0}^{\infty}\gamma_{ji}^{(0)}\big(\Phi^2(x_i)+|x_{i-j}|^{4m-2}\big) \\
&\le \frac{1}{2}\sum_{j=0}^{\infty}\gamma_{ji}^{(0)}\bigg(\Phi^2(x_i)+\frac{\Phi^2(x_{i-j})}{c^2}\bigg) \\
&= \frac{1}{2}\Bigg[\bigg(\rho_i^{(0)}+\frac{\gamma_{0i}^{(0)}}{c^2}\bigg)\Phi^2(x_i)+\frac{1}{c^2}\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}\Phi^2(x_{i-j})\Bigg].
\end{aligned} \tag{7.153}$$

Substituting (7.150)–(7.153) into (7.149) yields

$$\begin{aligned}
\Delta V_{1i} \le{}& -2(a_i-\kappa_i)\big(x_i+a_i\Phi(x_i)\big)\Phi(x_i) \\
&+(a_i-\kappa_i)^2\Phi^2(x_i)+2\sqrt{\delta_i}\big[x_i\Phi(x_i)+|\kappa_i|\Phi^2(x_i)\big] \\
&+\frac{1}{mc}\Bigg[\big(\rho_i^{(0)}+(2m-1)\gamma_{0i}^{(0)}\big)x_i\Phi(x_i)+(2m-1)\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}x_{i-j}\Phi(x_{i-j})\Bigg] \\
&+(1+\alpha_i)\delta_i\Phi^2(x_i)+\big(1+\alpha_i^{-1}\big)\frac{\rho_i^{(0)}}{c^2}\Bigg(\gamma_{0i}^{(0)}\Phi^2(x_i)+\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}\Phi^2(x_{i-j})\Bigg)
\end{aligned}$$

$$+\,\eta_{i+1}\left(\nu_i+\sum_{j=0}^{\infty}\gamma_{ji}^{(1)}x_{i-j}\Phi(x_{i-j})+\sum_{j=0}^{\infty}\gamma_{ji}^{(2)}\Phi^2(x_{i-j})\right)$$
$$+\,|\kappa_i|\left[\left(\rho_i^{(0)}+\frac{\gamma_{0i}^{(0)}}{c^2}\right)\Phi^2(x_i)+\frac{1}{c^2}\sum_{j=1}^{\infty}\gamma_{ji}^{(0)}\Phi^2(x_{i-j})\right]+\zeta_{i+1}.$$

Now, rewrite this inequality by summing over all terms involving $x_i\Phi(x_i)$ and $\Phi^2(x_i)$, respectively. This leads to

$$\begin{aligned}\Delta V_{1i}\le{}&\left[-2\big(a_i-\kappa_i-\sqrt{\delta_i}\big)+\eta_{i+1}\gamma_{0i}^{(1)}+\frac{1}{mc}\big(\rho_i^{(0)}+(2m-1)\gamma_{0i}^{(0)}\big)\right]x_i\Phi(x_i)\\&+\left[-\big(a_i^2-\kappa_i^2\big)+|\kappa_i|\left(2\sqrt{\delta_i}+\rho_i^{(0)}+\frac{\gamma_{0i}^{(0)}}{c^2}\right)\right.\\&\left.+\,\eta_{i+1}\gamma_{0i}^{(2)}+(1+\alpha_i)\delta_i+\big(1+\alpha_i^{-1}\big)\frac{\rho_i^{(0)}\gamma_{0i}^{(0)}}{c^2}\right]\Phi^2(x_i)\\&+\sum_{j=1}^{\infty}\beta_{ji}^{(1)}x_{i-j}\Phi(x_{i-j})+\sum_{j=1}^{\infty}\beta_{ji}^{(2)}\Phi^2(x_{i-j})\zeta_{i+1}+\nu_i\eta_{i+1},\end{aligned}$$

where

$$\begin{aligned}\beta_{ji}^{(1)}&=\gamma_{ji}^{(1)}\eta_{i+1}+\gamma_{ji}^{(0)}\frac{2m-1}{mc},\\\beta_{ji}^{(2)}&=\gamma_{ji}^{(2)}\eta_{i+1}+\gamma_{ji}^{(0)}\big(|\kappa_i|+\big(1+\alpha_i^{-1}\big)\rho_i^{(0)}\big)\frac{1}{c^2}.\end{aligned}\tag{7.154}$$

Now, set

$$V_{2i}=\sum_{j=1}^{\infty}\alpha_{j,i-1}^{(1)}x_{i-j}\Phi(x_{i-j})+\sum_{j=1}^{\infty}\alpha_{j,i-1}^{(2)}\Phi^2(x_{i-j}),$$

with

$$\alpha_{ji}^{(l)}=\sum_{k=j}^{\infty}\beta_{ki}^{(l)},\quad l=1,2.\tag{7.155}$$

From Lemma 7.4 it follows that

$$\begin{aligned}\Delta V_{2i}={}&\alpha_{1i}^{(1)}x_i\Phi(x_i)+\alpha_{1i}^{(2)}\Phi^2(x_i)\\&+\sum_{j=1}^{\infty}\left(\sum_{l=j+1}^{\infty}\big(\beta_{li}^{(1)}-\beta_{l,i-1}^{(1)}\big)-\beta_{j,i-1}^{(1)}\right)x_{i-j}\Phi(x_{i-j})\\&+\sum_{j=1}^{\infty}\left(\sum_{l=j+1}^{\infty}\big(\beta_{li}^{(2)}-\beta_{l,i-1}^{(2)}\big)-\beta_{j,i-1}^{(2)}\right)\Phi^2(x_{i-j}).\end{aligned}$$

As a result, for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\begin{aligned}\Delta V_i \le \Big[&-2\big(a_i - \kappa_i - \sqrt{\delta_i}\big) + \eta_{i+1}\gamma_{0i}^{(1)} \\ &+ \frac{1}{mc}\big(\rho_i^{(0)} + (2m-1)\gamma_{0i}^{(0)}\big) + \alpha_{1i}^{(1)}\Big] x_i \Phi(x_i) \\ &+ \bigg[-\big(a_i^2 - \kappa_i^2\big) + |\kappa_i|\bigg(2\sqrt{\delta_i} + \rho_i^{(0)} + \frac{\gamma_{0i}^{(0)}}{c^2}\bigg) \\ &+ \eta_{i+1}\gamma_{0i}^{(2)} + (1+\alpha_i)\delta_i + \big(1+\alpha_i^{-1}\big)\frac{\rho_i^{(0)}\gamma_{0i}^{(0)}}{c^2} + \alpha_{1i}^{(2)}\bigg]\Phi^2(x_i) \\ &+ \sum_{j=1}^{\infty}\bigg(\sum_{l=j}^{\infty}\big(\beta_{li}^{(1)} - \beta_{l,i-1}^{(1)}\big)\bigg) x_{i-j}\Phi(x_{i-j}) \\ &+ \sum_{j=1}^{\infty}\bigg(\sum_{l=j}^{\infty}\big(\beta_{li}^{(2)} - \beta_{l,i-1}^{(2)}\big)\bigg)\Phi^2(x_{i-j}) + \nu_i\eta_{i+1} + \zeta_{i+1}.\end{aligned}$$

Via the representations (7.141) for $\varepsilon_i^{(1)}$ and (7.154) and (7.155) for $\alpha_{1i}^{(1)}$ and $\beta_{ji}^{(1)}$ we obtain

$$\begin{aligned}\Delta V_i \le -\varepsilon_i^{(1)} x_i \Phi(x_i) + \bigg[&-\big(a_i^2 - \kappa_i^2\big) + |\kappa_i|\bigg(2\sqrt{\delta_i} + \bigg(1 + \frac{1}{c^2}\bigg)\rho_i^{(0)}\bigg) \\ &+ \eta_{i+1}\rho_i^{(2)} + (1+\alpha_i)\delta_i + \big(1+\alpha_i^{-1}\big)\bigg(\frac{\rho_i^{(0)}}{c}\bigg)^2\bigg]\Phi^2(x_i) \\ &+ \sum_{j=1}^{\infty}\bigg(\sum_{l=j}^{\infty}\big(\beta_{li}^{(1)} - \beta_{l,i-1}^{(1)}\big)\bigg) x_{i-j}\Phi(x_{i-j}) \\ &+ \sum_{j=1}^{\infty}\bigg(\sum_{l=j}^{\infty}\big(\beta_{li}^{(2)} - \beta_{l,i-1}^{(2)}\big)\bigg)\Phi^2(x_{i-j}) + \nu_i\eta_{i+1} + \zeta_{i+1}.\end{aligned}$$

Via Lemma 1.2 to minimize right hand part of this inequality put $\alpha_i = \rho_i^{(0)}(c\sqrt{\delta_i})^{-1}$. With this α_i the representation (7.154) coincides with (7.144). Via the hypothesis (H5), the $\beta_{ji}^{(l)}$ are non-increasing in $i \in Z$. Thus, using (7.142), we get the estimate

$$\Delta V_i \le -\varepsilon_i^{(1)} x_i \Phi(x_i) - \varepsilon_i^{(2)}\Phi^2(x_i) + \nu_i\eta_{i+1} + \zeta_{i+1}, \quad i \in Z.$$

By summing this inequality over i, we obtain the decomposition

$$V_i \le V_0 + A_i^{(1)} - A_i^{(2)} + M_i, \quad i \in Z,$$

with

$$A_i^{(1)} = \sum_{j=0}^{i-1} \nu_j \eta_{j+1}, \qquad A_i^{(2)} = \sum_{j=0}^{i-1} \big(\varepsilon_j^{(1)} x_j \Phi(x_j) + \varepsilon_j^{(2)} \Phi^2(x_j)\big), \qquad M_i = \sum_{j=1}^{i} \zeta_j.$$

Note that $\lim_{i\to\infty} A_i^{(1)} < \infty$ (a.s.) due to the condition (7.140) in the hypothesis (H2). Eventually, we may apply Lemma 7.3 to the sequence $W_i := V_0 + A_i^{(1)} - A_i^{(2)} + M_i$. In fact, $W_i \geq V_i$ is positive and all assumptions of Lemma 7.3 are satisfied. Therefore, W_i converges (a.s.) to a finite limit $W_\infty = \lim_{i\to\infty} W_i$ and $A_i^{(2)}$ also converges to the finite limit $A_\infty^{(2)} = \lim_{i\to\infty} A_i^{(2)}$ (a.s.). In particular, this implies that V_i is a positive, bounded sequence (a.s.) for all finite values V_0. By construction of V_i and $y_i^2 = (x_i + a_i \Phi(x_i))^2$, the y_i must satisfy the condition

$$0 \leq \limsup_{i\to\infty} y_i^2 < \infty \quad \text{(a.s.)}.$$

Note that $y_i^2 \geq x_i^2 + a_i^2 \Phi^2(x_i)$ under (H1). Hence, we can easily conclude that

$$0 \leq \limsup_{i\to\infty} x_i^2 < \infty \quad \text{(a.s.)}.$$

Let us prove that $\lim_{i\to\infty} x_i^2 = 0$. Suppose, indirectly, that the opposite is true. Then there exists a.s. a finite $c_0^2(\omega) > 0$ on $\Omega_1 = \{\omega : \limsup_{i\to\infty} x_i^2(\omega) = c_0^2(\omega) > 0\}$ with $\mathbf{P}(\Omega_1) = p_1 > 0$. There also exists a subsequence x_{i_k}, $i_k \in Z$, an a.s. finite random variable $c_1 = c_1(\omega) > 0$ and an integer $N(\omega)$ such that

$$\big|x_{i_k}(\omega)\big| \geq c_0(\omega), \qquad x_{i_k}\Phi(x_{i_k}) \geq c_1(\omega), \qquad \Phi^2(x_{i_k}) \geq c_1(\omega), \tag{7.156}$$

for all $i_k \geq N(\omega)$ on $\omega \in \Omega_1$. Let $I_N^i = \{i_k \in Z : i \geq i_k \geq N(\omega), \text{(7.156) holds}\}$. Note that the cardinality $\#(I_N^n)$ of the set I_N^i tends to ∞ as $i \to \infty$. Then, for all $\omega \in \Omega_1$ and for all $i > \max\{N(\omega), i_0\}$, we have

$$\begin{aligned}
A_i^{(2)}(\omega) &= \sum_{j=0}^{i} \big(\varepsilon_j^{(1)} x_j \Phi(x_j) + \varepsilon_j^{(2)} \Phi^2(x_j)\big) \\
&\geq \sum_{j=N}^{i} \big(\varepsilon_j^{(1)} x_j \Phi(x_j) + \varepsilon_j^{(2)} \Phi^2(x_j)\big) \\
&\geq \sum_{j=N, j\in I_N^i}^{i} \big(\varepsilon_j^{(1)} x_j \Phi(x_j) + \varepsilon_j^{(2)} \Phi^2(x_j)\big) \\
&\geq c_1(\omega) \sum_{j=N, j\in I_N^i}^{i} \big(\varepsilon_j^{(1)} + \varepsilon_j^{(2)}\big) \geq c_1(\omega)\varepsilon \#\big(I_N^n\big) \to \infty
\end{aligned}$$

as $i \to \infty$, due to the hypothesis (H4). Therefore, $\limsup_{i\to\infty} A_i^{(2)} = \lim_{i\to\infty} A_i^{(2)} = \infty$. This result contradicts the finiteness of $\lim_{i\to\infty} A_i^{(2)}$ as claimed by Lemma 7.3. Therefore, the condition $\lim_{i\to\infty} x_i = 0$ holds (a.s.), independently of the initial values φ_j, $j \in Z_0$. This observation obviously confirms the conclusion of Theorem 7.6. □

Remark 7.15 Note that Remark 7.14 implies that Theorem 7.6 holds true if there exists a non-random $i_0 \in Z$ such that conditions (7.136), (7.141), (7.142) and (7.144) are fulfilled for all $i \geq i_0$.

Theorem 7.7 *Let* ξ_i, $i \in Z$, *be square-integrable, independent random variables with* $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = \eta_i$ *and let the hypotheses* (H0)–(H3) *and* (H5) *be fulfilled and*

$$\sum_{i=1}^{\infty}\bigl(\varepsilon_i^{(1)} + \varepsilon_i^{(2)}\bigr) = \infty. \tag{7.157}$$

Suppose, in addition, that the coefficients $\gamma_{ji}^{(l)}$ *from conditions* (7.138) *and* (7.139) *possess the following property: there exists* $k \in Z$ *such that for all* $l = 0, 1, 2$

$$\begin{aligned} &\lim_{i\to\infty} \gamma_{ji}^{(l)} = 0, \quad j = 1, 2, \dots, k, \\ &\gamma_{ji}^{(l)} = 0, \quad j > k,\ i \in Z. \end{aligned} \tag{7.158}$$

Assume also that one of the following conditions holds:

(i) $\lim_{i\to\infty} a_i = 0$.
(ii) $\lim_{i\to\infty} a_i = a > 0$, *function* $\Phi(x)$ *does not decrease and* $\Phi(-x) = -\Phi(x)$ *for all* $x \in \mathbf{R}$.

Then the trivial solution of (7.133) *is globally a.s. asymptotically stable.*

Remark 7.16 Note that the function $\Phi(x) = \sum_{j=1}^{m} \alpha_j x^{2j-1}$, $\alpha_j \geq 0$, $m \geq 1$, satisfies the conditions stated in (ii); in particular, $\Phi(x) = x^{2m-1}$ does.

Proof In the same way as in Theorem 7.6, we prove that $\lim_{i\to\infty} V_i$ exists. Then for some a.s. finite random variable $H(\omega) > 0$ and all $i \in Z$ we have $x_i\Phi(x_i) \leq H$, $\Phi^2(x_i) \leq H$. We show that $\lim_{i\to\infty} V_{1i}$ exists. We note first that via (7.154) and (7.158)

$$\begin{aligned} V_{2i} &= \sum_{j=1}^{\infty}\bigl(\alpha_{j,i-1}^{(1)} x_{i-j}\Phi(x_{i-j}) + \alpha_{j,i-1}^{(2)}\Phi^2(x_{i-j})\bigr) \\ &= \sum_{j=1}^{\infty}\sum_{l=j}^{k}\bigl(\beta_{l,i-1}^{(1)} x_{i-j}\Phi(x_{i-j}) + \beta_{l,i-1}^{(2)}\Phi^2(x_{iji})\bigr) \end{aligned}$$

$$
\le H \sum_{j=1}^{k} \sum_{l=j}^{k} \left(\beta_{l,i-1}^{(1)} + \beta_{l,i-1}^{(2)}\right)
$$

$$
\le H \frac{k(k+1)}{2} \max_{j=1,\dots,k} \left\{\beta_{j,i-1}^{(1)} + \beta_{j,i-1}^{(2)}\right\} \to 0
$$

when $i \to \infty$. The above relation implies also that $\lim_{i\to\infty} V_{1i} = 0$. To prove that $\lim_{i\to\infty} x_i^2$ exists, we suppose the opposite, i.e. there are numbers $\bar{l}$, $\underline{l}$ with $|\bar{l}| > |\underline{l}|$ and sequences i_k, $k \in Z$, i_l, $l \in Z$, such that

$$
\lim_{k\to\infty} x_{i_k} = \bar{l}, \qquad \lim_{l\to\infty} x_{i_l} = \underline{l}.
$$

By substituting i_k and i_l instead of i in the expression $V_{1i} = (x_i + a_i \Phi(x_i))^2$ and passing to the limit twice, we arrive at

$$
\left(\bar{l} + a\Phi\left(\bar{l}\right)\right)^2 = \left(\underline{l} + a\Phi\left(\underline{l}\right)\right)^2. \tag{7.159}
$$

In case (i) equality (7.159) is reduced to $\bar{l}^2 = \underline{l}^2$, which contradicts the assumption $|\bar{l}| > |\underline{l}|$.

In case (ii) the equality (7.159) is also impossible. Note that $\Psi(x) = x + a\Phi(x)$ is a strictly increasing function. When $\operatorname{sign}\bar{l} = \operatorname{sign}\underline{l}$, the equality (7.159) implies that $\Psi(\bar{l}) = \Psi(\underline{l})$, which is impossible since $\bar{l} \neq \underline{l}$. Suppose now that $\bar{l} \geq 0 \geq \underline{l}$. Then $\bar{l} + a\Phi(\bar{l}) = -\underline{l} - a\Phi(\underline{l})$. Since $-\underline{l} - a\Phi(\underline{l}) = -\underline{l} + a\Phi(-\underline{l}) = \Psi(-\underline{l})$, we arrive at $\Psi(-\underline{l}) = \Psi(\bar{l})$, which is also impossible, since $-\underline{l}$ cannot be equal to $\bar{l}$. The case $\bar{l} \leq 0 \leq \underline{l}$ can be treated in the same way.

Thus, $\lim_{i\to\infty} x_i^2$ exists. In order to prove that $\lim_{i\to\infty} x_i = 0$, we act in a similar way as in the proof of Theorem 7.6. □

Chapter 8
Volterra Equations of Second Type

In this chapter the asymptotic behavior of the solutions of stochastic difference second kind Volterra equations is studied by virtue of the procedure of the construction of Lyapunov functionals and also by the resolvent representation of the solution.

8.1 Statement of the Problem

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be the basic probability space, $i \in Z = \{0, 1, \dots\}$ discrete time, $\mathfrak{F}_i \in \mathfrak{F}$ a nondecreasing family of σ-algebras, H_p, $p > 0$, the space of sequences $x = \{x_i, i \in Z\}$, $\mathfrak{F}_i$-adapted random values $x_i \in \mathbf{R}^n$ with the norm $\|x\|^p = \sup_{i\in Z} \mathbf{E}|x_i|^p$, $\eta = \{\eta_i, i \in Z\} \in H_p$.

Consider the stochastic difference equation in the form

$$\begin{aligned} x_{i+1} &= \eta_{i+1} + F(i, x_0, \dots, x_i), \quad i \in Z, \\ x_0 &= \eta_0, \end{aligned} \tag{8.1}$$

and the auxiliary difference equation

$$\begin{aligned} x_{i+1} &= F(i, x_0, \dots, x_i), \quad i \in Z, \\ x_0 &= \eta_0. \end{aligned} \tag{8.2}$$

It is supposed that the functional F in (8.1) and (8.2) is such that $F : Z * H_p \Rightarrow R^n$ and $F(i, \cdot)$ does not depend on x_j for $j > i$, $F(i, 0, \dots, 0) = 0$, $\eta \in H_p$.

Definition 8.1 The sequence x_i from H_p, $p > 0$, is called:

– Uniformly p-bounded if $\|x\|^p < \infty$.
– Asymptotically p-trivial if $\lim_{i\to\infty} \mathbf{E}|x_i|^p = 0$.
– p-summable if $\sum_{i=0}^{\infty} \mathbf{E}|x_i|^p < \infty$.

Definition 8.2 The solution $x = \{x_0, x_1, \dots\}$ of (8.1) is called p-stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x\|^p < \varepsilon$ if $\|\eta\|^p < \delta$.

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_8,

In particular, if $p = 1$ then x_i is called, respectively, uniformly mean bounded, asymptotically mean trivial, mean summable and mean stable. If $p = 2$ then x_i is called correspondingly uniformly mean square bounded, asymptotically mean square trivial, mean square summable and mean square stable.

Remark 8.1 It is easy to see that if the sequence x_i is p-summable, then it is uniformly p-bounded and asymptotically p-trivial.

Theorem 8.1 *Let there exist a nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ and a sequence of nonnegative numbers γ_i such that*

$$\mathbf{E}V(0, x_0) < \infty, \qquad \sum_{i=0}^{\infty} \gamma_i < \infty. \tag{8.3}$$

$$\mathbf{E}\Delta V_i \le -c\mathbf{E}|x_i|^p + \gamma_i, \quad i \in Z, \; c > 0. \tag{8.4}$$

Then the solution of (8.1) *is p-summable.*

Proof From (8.4) it follows that

$$\sum_{j=0}^{i} \mathbf{E}\Delta V_j = \mathbf{E}V(i+1, x_0, \dots, x_{i+1}) - \mathbf{E}V(0, x_0) \le -c\sum_{j=0}^{i} \mathbf{E}|x_j|^p + \sum_{j=0}^{i} \gamma_j.$$

From this by virtue of (8.3) we obtain

$$c\sum_{j=0}^{i} \mathbf{E}|x_j|^p \le \mathbf{E}V(0, x_0) + \sum_{j=0}^{\infty} \gamma_j < \infty.$$

Therefore, the solution of (8.1) is p-summable. Theorem is proven. □

Corollary 8.1 *Let there exist a nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ and a sequence of nonnegative numbers γ_i such that the conditions* (8.3) *hold and*

$$\begin{gathered} \mathbf{E}\Delta V_i \le -c\mathbf{E}|x_i|^p + \sum_{j=0}^{i} A_{ij}\mathbf{E}|x_j|^p + \gamma_i, \\ A_{ij} \ge 0, \quad i \in Z, \; j = 0, \dots, i, \; \sup_{j \in Z} \sum_{i=j}^{\infty} A_{ij} < c. \end{gathered} \tag{8.5}$$

Then the solution of (8.1) *is p-summable.*

The proof follows from Theorem 8.1 and the proof of Theorem 1.2.

From Theorem 8.1 and Corollary 8.1 it follows that investigation of solution asymptotic behavior of stochastic difference equations type of (8.1) can be reduced to the construction of appropriate Lyapunov functionals. For this aim the formal procedure of the construction of Lyapunov functionals which was described in Sect. 1.2 can be used by the assumption that $G(i, j, \dots) = 0$. Note also that Theorem 1.1 is applicable for (8.2).

Below, this procedure is demonstrated for one simple scalar equation.

8.2 Illustrative Example

Using the procedure of the construction of Lyapunov functionals let us investigate the asymptotic behavior of solution of the scalar equation with constant coefficients

$$\begin{aligned} &x_0 = \eta_0, \qquad x_1 = \eta_1 + a_0\eta_0, \\ &x_{i+1} = \eta_{i+1} + a_0 x_i + a_1 x_{i-1}, \quad i \geq 1. \end{aligned} \tag{8.6}$$

8.2.1 First Way of the Construction of the Lyapunov Functional

The right-hand side of (8.6) is already represented in the form (1.7) with $\tau = 0$,

$$F_1(i, x_i) = a_0 x_i, \qquad F_2(i, x_0, \dots, x_i) = a_1 x_{i-1}, \qquad F_3(i, x_0, \dots, x_i) = 0.$$

Auxiliary difference equation (1.8) in this case is $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0| < 1$, since $\Delta v_i = (a_0^2 - 1)y_i^2$.

Put $V_i = x_i^2$. Calculating $\mathbf{E}\Delta V_i$ for (8.6) and using some $\lambda > 0$ we get

$$\begin{aligned} \mathbf{E}\Delta V_i &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = \mathbf{E}\big[(\eta_{i+1} + a_0 x_i + a_1 x_{i-1})^2 - x_i^2\big] \\ &\leq \big[1 + \lambda^{-1}\big(|a_0| + |a_1|\big)\big]\mathbf{E}\eta_{i+1}^2 + \big(a_0^2 - 1 + |a_0 a_1| + \alpha|a_0|\big)\mathbf{E}x_i^2 \\ &\quad + \big(a_1^2 + |a_0 a_1| + \lambda|a_1|\big)\mathbf{E}x_{i-1}^2. \end{aligned}$$

If

$$|a_0| + |a_1| < 1 \tag{8.7}$$

then there exists small enough $\lambda > 0$ such that

$$\big(|a_0| + |a_1|\big)^2 + \lambda\big(|a_0| + |a_1|\big) < 1.$$

Thus, if the condition (8.7) holds and the sequence η_i, $i \in Z$, is mean square summable then the functional V_i satisfies the conditions of Corollary 8.1 by $p = 2$ and therefore the solution of (8.6) is mean square summable.

Note that the summability region corresponding to the condition (8.7) is shown in Fig. 2.1 (number 1) by $a = a_0$, $b = a_1$.

8.2.2 Second Way of the Construction of the Lyapunov Functional

Let us use another representation of (8.6). Represent the right-hand side of (8.6) in the form (1.7) with $\tau = 0$,

$$F_1(i, x_i) = (a_0 + a_1)x_i, \qquad F_2(i, x_0, \dots, x_i) = 0,$$
$$F_3(i, x_0, \dots, x_i) = -a_1 x_{i-1}.$$

Auxiliary difference equation (1.8) in this case has the form $y_{i+1} = (a_0 + a_1)y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0 + a_1| < 1$, since $\Delta v_i = ((a_0 + a_1)^2 - 1)y_i^2$.

Put $V_i = (x_i + a_1 x_{i-1})^2$. Calculating $\mathbf{E}\Delta V_i$ for (8.6) and using some $\lambda > 0$ we get

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big[(x_{i+1} + a_1 x_i)^2 - (x_i + a_1 x_{i-1})^2\big] \\
&= \mathbf{E}\big(\eta_{i+1} + (a_0 + a_1 - 1)x_i\big)\big(\eta_{i+1} + (a_0 + a_1 + 1)x_i + 2a_1 x_{i-1}\big) \\
&\le \big[1 + \lambda^{-1}\big(|a_1| + |a_0 + a_1|\big)\big]\mathbf{E}\eta_{i+1}^2 \\
&\quad + \big[(a_0 + a_1)^2 - 1 + \big|a_1(a_0 + a_1 - 1)\big| + \lambda|a_0 + a_1|\big]\mathbf{E}x_i^2 \\
&\quad + \big(\big|a_1(a_0 + a_1 - 1)\big| + \lambda|a_1|\big)\mathbf{E}x_{i-1}^2.
\end{aligned}$$

If

$$(a_0 + a_1)^2 + 2\big|a_1(a_0 + a_1 - 1)\big| < 1 \tag{8.8}$$

then there exists a small enough $\lambda > 0$ so that

$$(a_0 + a_1)^2 + 2\big|a_1(a_0 + a_1 - 1)\big| + \lambda\big(|a_1| + |a_0 + a_1|\big) < 1.$$

Thus, if the condition (8.8) holds and the sequence η_i, $i \in Z$, is mean square summable then the functional V_i satisfies the conditions of Corollary 8.1 by $p = 2$ and therefore the solution of (8.6) is mean square summable.

Note that the condition (8.8) can be rewritten in the form

$$|a_0 + a_1| < 1, \quad 2|a_1| < 1 + a_0 + a_1.$$

The summability region corresponding to the condition (8.8) is shown in Fig. 2.2 (number 1) by $a = a_0$, $b = a_1$.

8.2.3 Third Way of the Construction the Lyapunov Functional

In some cases the auxiliary equation can be obtained by iterating the right-hand side of (8.6). For example, from (8.6) we get

$$x_0=\eta_0,\qquad x_1=\eta_1+a_0\eta_0,\qquad x_2=\eta_2+a_0\eta_1+\left(a_0^2+a_1\right)x_0,$$
$$x_{i+1}=\eta_{i+1}+a_0\eta_i+\left(a_0^2+a_1\right)x_{i-1}+a_0a_1x_{i-2},\quad i\ge 2.$$

Let $\tau=0$,

$$F_1(i,x_i)=F_3(i,x_0,\dots,x_i)=0,\qquad F_2(i,x_0,\dots,x_i)=\left(a_0^2+a_1\right)x_{i-1}+a_0a_1x_{i-2}.$$

The auxiliary difference equation is $y_{i+1}=0$, $i\ge 0$. The function $v_i=y_i^2$ is a Lyapunov function for this equation since $\Delta v_i=y_{i+1}^2-y_i^2=-y_i^2$.

Put $V_i=x_i^2$. Calculating $\mathbf{E}\Delta V_i$ and using some $\lambda>0$ we get

$$\begin{aligned}
\mathbf{E}\Delta V_i&=\mathbf{E}\left(x_{i+1}^2-x_i^2\right)\\
&=\mathbf{E}\left[\left(\eta_{i+1}+a_0\eta_i+\left(a_0^2+a_1\right)x_{i-1}+a_0a_1x_{i-2}\right)^2-x_i^2\right]\\
&\le -\mathbf{E}x_i^2+\mathbf{E}\Big[\eta_{i+1}^2+a_0^2\eta_i^2+\left(a_0^2+a_1\right)^2x_{i-1}^2+a_0^2a_1^2x_{i-2}^2\\
&\quad+|a_0|\left(\eta_{i+1}^2+\eta_i^2\right)+\left|a_0^2+a_1\right|\left(\lambda^{-1}\eta_{i+1}^2+\lambda x_{i-1}^2\right)\\
&\quad+|a_0a_1|\left(\lambda^{-1}\eta_{i+1}^2+\lambda x_{i-2}^2\right)+\left|a_0\left(a_0^2+a_1\right)\right|\left(\lambda^{-1}\eta_i^2+\lambda x_{i-1}^2\right)\\
&\quad+\left|a_0^2a_1\right|\left(\lambda^{-1}\eta_i^2+\lambda x_{i-2}^2\right)+\left|a_0a_1\left(a_0^2+a_1\right)\right|\left(x_{i-1}^2+x_{i-2}^2\right)\Big]\\
&=-\mathbf{E}x_i^2+A_1\mathbf{E}x_{i-1}^2+A_2\mathbf{E}x_{i-2}^2+\gamma_i,
\end{aligned}$$

where

$$\gamma_i=A_0\left(\mathbf{E}\eta_{i+1}^2+|a_0|\mathbf{E}\eta_i^2\right),$$
$$A_0=1+|a_0|+\lambda^{-1}\left(\left|a_0^2+a_1\right|+|a_0a_1|\right),$$
$$A_1=\left(a_0^2+a_1\right)^2+\left|a_0a_1\left(a_0^2+a_1\right)\right|+\lambda\left(1+|a_0|\right)\left|a_0^2+a_1\right|,$$
$$A_2=a_0^2a_1^2+\left|a_0a_1\left(a_0^2+a_1\right)\right|+\lambda\left(1+|a_0|\right)|a_0a_1|.$$

If

$$\left|a_0^2+a_1\right|+|a_0a_1|<1 \tag{8.9}$$

then there exists a small enough $\lambda>0$ so that

$$A_1+A_2=\left(\left|a_0^2+a_1\right|+|a_0a_1|\right)^2+\lambda\left(1+|a_0|\right)\left(\left|a_0^2+a_1\right|+|a_0a_1|\right)<1.$$

Thus, if the condition (8.9) holds and the sequence η_i, $i\in Z$, is mean square summable then the functional V_i satisfies the conditions of Corollary 8.1 by $p=2$ and therefore the solution of (8.6) is mean square summable.

Note that the summability region corresponding to the condition (8.9) is shown in Fig. 2.3 (number 1) by $a = a_0$, $b = a_1$.

8.2.4 Fourth Way of the Construction of the Lyapunov Functional

Consider now the case $\tau = 1$. Represent (8.6) in the form (1.7) by $F_1(i, x_{i-1}, x_i) = a_0 x_i + a_1 x_{i-1}$, $F_2(i, x_0, \dots, x_i) = F_3(i, x_0, \dots, x_i) = 0$. In this case the auxiliary difference equation is

$$y_{i+1} = a_0 y_i + a_1 y_{i-1}. \tag{8.10}$$

Introduce the vector $y(i) = (y_{i-1}, y_i)'$. Then (8.10) can be represented in the form

$$y(i+1) = Ay(i), \quad A = \begin{pmatrix} 0 & 1 \\ a_1 & a_0 \end{pmatrix}. \tag{8.11}$$

Let the matrix D be a solution of the equation $A'DA - D = -U$, where $U = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the matrix D has (see (2.14)) the elements d_{ij} such that

$$\begin{aligned} d_{11} &= a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1 d_{22}}{1 - a_1}, \\ d_{22} &= \frac{(1 - a_1)}{(1 + a_1)[(1 - a_1)^2 - a_0^2]}. \end{aligned} \tag{8.12}$$

The matrix D is a positively semidefinite matrix by the conditions

$$|a_1| < 1, \quad |a_0| < 1 - a_1. \tag{8.13}$$

Put $v_i = y'(i)Dy(i)$. Then

$$\begin{aligned} \Delta v_i &= y'(i+1)Dy(i+1) - y'(i)Dy(i) = y'(i)\big[A'DA - D\big]y(i) \\ &= -y'(i)Uy(i) = -y_i^2. \end{aligned}$$

Thus, the function $v_i = y'(i)Dy(i)$ under the conditions (8.13) is a Lyapunov function for (8.10).

Put $x(i) = (x_{i-1}, x_i)'$, $\eta(i) = (0, \eta_i)'$. Then

$$x(i+1) = \eta(i+1) + Ax(i).$$

Putting $V_i = x'(i)Dx(i)$ and calculating $\mathbf{E}\Delta V_i$, we get

$$\begin{aligned} \mathbf{E}\Delta V_i &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big] \\ &= -\mathbf{E}x'(i)Ux(i) + \mathbf{E}\eta'(i+1)D\eta(i+1) + 2\mathbf{E}\eta'(i+1)DAx(i) \\ &= -\mathbf{E}x_i^2 + d_{22}\mathbf{E}\eta_{i+1}^2 + 2(d_{12} + d_{22}a_0)\mathbf{E}x_i\eta_{i+1} + 2d_{22}a_1\mathbf{E}x_{i-1}\eta_{i+1}. \end{aligned}$$

Via (8.12) and (8.13)

$$|d_{12}+d_{22}a_0| = d_{22}\frac{|a_0|}{1-a_1} < d_{22}, \quad |d_{22}a_1| < d_{22}.$$

Therefore,

$$\mathbf{E}\Delta V_i \le -\mathbf{E}x_i^2 + d_{22}\mathbf{E}\eta_{i+1}^2 + 2d_{22}\sum_{j=i-1}^{i}\mathbf{E}|x_j\eta_{i+1}|.$$

For arbitrary $\lambda > 0$ we have

$$2|x_j\eta_{i+1}| \le \frac{\lambda}{d_{22}}x_j^2 + \frac{d_{22}}{\lambda}\eta_{i+1}^2.$$

Thus,

$$\mathbf{E}\Delta V_i \le -(1-\lambda)\mathbf{E}x_i^2 + \lambda\mathbf{E}x_{i-1}^2 + d_{22}\big(1+2\lambda^{-1}d_{22}\big)\mathbf{E}\eta_{i+1}^2.$$

It means that if the condition (8.13) holds and the sequence η_i, $i \in Z$, is mean square summable then for small enough $\lambda > 0$ such that $2\lambda < 1$ the functional V_i satisfies the conditions of Corollary 8.1 by $p = 2$ and therefore the solution of (8.6) is mean square summable.

Note that the summability region corresponding to the conditions (8.13) is shown in Fig. 2.4 (number 1) by $a = a_0$, $b = a_1$.

8.3 Linear Equations with Constant Coefficients

Consider the difference equation

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} a_{i-j}x_j, \quad i \in Z,\ x_0 = \eta_0. \tag{8.14}$$

Here a_i are known constants.

Let us apply the proposed procedure of the construction of Lyapunov functionals to this equation.

8.3.1 First Way of the Construction of the Lyapunov Functional

Represent the right-hand side of (8.14) in the form (1.7) with $\tau = 0$,

$$F_1(i, x_i) = a_0x_i, \qquad F_2(i, x_0, \dots, x_i) = \sum_{l=0}^{i-1} a_{i-l}x_l, \qquad F_3(i, x_0, \dots, x_i) = 0.$$

Auxiliary difference equation (1.8) in this case is $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ can be taken as a Lyapunov function for this equation if $|a_0| < 1$ since $\Delta v_i = (a_0^2 - 1)y_i^2$.

Put now $V_i = x_i^2$ and

$$\alpha_1 = \sum_{l=0}^{\infty} |a_l|, \quad A_j = (\lambda + \alpha_1)|a_j|, \quad j \geq 0, \ \lambda > 0.$$

Estimating $\mathbf{E}\Delta V_i$, we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = -\mathbf{E}x_i^2 + \mathbf{E}\Bigg(\eta_{i+1} + \sum_{j=0}^{i} a_{i-j}x_j\Bigg)^2 \\
&= -\mathbf{E}x_i^2 + \mathbf{E}\eta_{i+1}^2 + \mathbf{E}\Bigg(\sum_{j=0}^{i} a_{i-j}x_j\Bigg)^2 + 2\sum_{j=0}^{i} a_{i-j}\mathbf{E}x_j\eta_{i+1} \\
&\leq -\mathbf{E}x_i^2 + \mathbf{E}\eta_{i+1}^2 + \sum_{l=0}^{i} |a_{i-l}| \sum_{j=0}^{i} |a_{i-j}|\mathbf{E}x_j^2 \\
&\quad + \sum_{j=0}^{i} |a_{i-j}|\big(\lambda^{-1}\mathbf{E}\eta_{i+1}^2 + \lambda\mathbf{E}x_j^2\big) \\
&\leq -\mathbf{E}x_i^2 + \sum_{j=0}^{i} A_{i-j}\mathbf{E}x_j^2 + \big(1 + \lambda^{-1}\alpha_1\big)\mathbf{E}\eta_{i+1}^2.
\end{aligned}$$

So, if

$$\alpha_1 < 1 \tag{8.15}$$

then there exists small enough $\lambda > 0$ such that $\sum_{i=0}^{\infty} A_i = \alpha_1(\lambda + \alpha_1) < 1$ and functional V_i satisfies the conditions of Corollary 8.1 by $p = 2$.

Therefore, if the condition (8.15) holds and the sequence η_i, $i \in Z$, is mean square summable then the solution of (8.6) is mean square summable.

In particular, for (8.6) we have $\alpha_1 = |a_0| + |a_1|$ and from (8.15) the condition (8.7) follows.

8.3.2 Second Way of the Construction of the Lyapunov Functional

Represent the right-hand side of (8.14) in the form (1.7) with $\tau = 0$,

$$F_1(i, x_i) = \beta x_i, \quad \beta = \sum_{j=0}^{\infty} a_j,$$

$$F_2(i, x_0, \dots, x_i) = 0, \qquad F_3(i, x_0, \dots, x_i) = -\sum_{l=0}^{i-1} x_l \sum_{j=i-l}^{\infty} a_j.$$

Auxiliary difference equation (1.8) in this case is $y_{i+1} = \beta y_i$. The function $v_i = y_i^2$ can be taken as Lyapunov function for this equation if $\beta < 1$, since $\Delta v_i = (\beta^2 - 1) y_i^2$.

Put $V_i = (x_i - F_{3i})^2$ and

$$\alpha = \sum_{l=1}^{\infty} \left| \sum_{m=l}^{\infty} a_m \right|.$$

Calculating $\mathbf{E}\Delta V_i$ via some $\lambda > 0$ we get

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big[(x_{i+1} - F_{3,i+1})^2 - (x_i - F_{3i})^2\big] \\
&= \mathbf{E}(x_{i+1} - F_{3,i+1} - x_i + F_{3i})(x_{i+1} - F_{3,i+1} + x_i - F_{3i}) \\
&= \mathbf{E}(\eta_{i+1} + \beta x_i + \Delta F_{3i} - F_{3,i+1} - x_i + F_{3i})(\eta_{i+1} \\
&\quad + \beta x_i + \Delta F_{3i} - F_{3,i+1} + x_i - F_{3i}) \\
&= \mathbf{E}\big(\eta_{i+1} + (\beta - 1)x_i\big)\left(\eta_{i+1} + (\beta + 1)x_i + 2\sum_{l=0}^{i-1} x_l \sum_{j=i-l}^{\infty} a_j\right) \\
&= \mathbf{E}\eta_{i+1}^2 + \big(\beta^2 - 1\big)\mathbf{E}x_i^2 + 2\beta \mathbf{E}x_i\eta_{i+1} + 2\sum_{l=0}^{i-1} \mathbf{E}\eta_{i+1}x_l \sum_{j=i-l}^{\infty} a_j \\
&\quad + 2(\beta - 1)\sum_{l=0}^{i-1} \mathbf{E}x_i x_l \sum_{j=i-l}^{\infty} a_j \\
&\le \mathbf{E}\eta_{i+1}^2 + \big(\beta^2 - 1\big)\mathbf{E}x_i^2 + |\beta|\big(\lambda \mathbf{E}x_i^2 + \lambda^{-1}\mathbf{E}\eta_{i+1}^2\big) \\
&\quad + \sum_{l=0}^{i-1} \left| \sum_{j=i-l}^{\infty} a_j \right| \big(\lambda \mathbf{E}x_l^2 + \lambda^{-1}\mathbf{E}\eta_{i+1}^2\big) + |\beta - 1| \sum_{l=0}^{i-1} \left| \sum_{j=i-l}^{\infty} a_j \right| \big(\mathbf{E}x_i^2 + \mathbf{E}x_l^2\big) \\
&\le \big(1 + \lambda^{-1}\big(|\beta| + \alpha\big)\big)\mathbf{E}\eta_{i+1}^2 + \big(\beta^2 - 1 + \lambda|\beta| + \alpha|\beta - 1|\big)\mathbf{E}x_i^2 \\
&\quad + \big(\lambda + |\beta - 1|\big) \sum_{l=0}^{i-1} \left| \sum_{j=i-l}^{\infty} a_j \right| \mathbf{E}x_l^2.
\end{aligned}$$

If

$$\beta^2 + 2\alpha|\beta - 1| < 1 \tag{8.16}$$

then there exists small enough $\lambda > 0$ such that $\beta^2 + 2\alpha|\beta - 1| + \lambda(|\beta| + \alpha) < 1$ and the functional V_i satisfies the conditions of Corollary 8.1 by $p = 2$.

Therefore, if the condition (8.16) holds and the sequence η_i, $i \in Z$, is mean square summable then the solution of (8.6) is mean square summable.

Note that the condition (8.16) can be rewritten in the form

$$|\beta| < 1, \quad 2\alpha < 1 + \beta.$$

In particular, for (8.14) we have $\beta = a_0 + a_1$, $\alpha = |a_1|$, and from (8.16) the condition (8.8) follows.

Example 8.1 Consider the equation

$$x_{i+1} = \eta_{i+1} + a_0 x_i + a_2 x_{i-2}. \tag{8.17}$$

From (8.15) and (8.16) we obtain two conditions for the mean square summability of the solution of (8.17):

$$|a_0| + |a_2| < 1$$

and

$$|a_0 + a_2| < 1, \quad 4|a_2| < 1 + a_0 + a_2.$$

For getting a third condition let us represent (8.17) in form (1.7) by $\tau = 2$,

$$F_1(i, x_{i-2}, x_{i-1}, x_i) = a_0 x_i + a_2 x_{i-2},$$
$$F_2(i, x_0, \dots, x_i) = F_3(i, x_0, \dots, x_i) = 0.$$

In this case the auxiliary difference equation is $y_{i+1} = a_0 y_i + a_2 y_{i-2}$. Put

$$y(i) = \begin{pmatrix} y_{i-2} \\ y_{i-1} \\ y_i \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_2 & 0 & a_0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the auxiliary difference equation can be represented in the form $y(i+1) = Ay(i)$ and the solution D of the matrix equation $A'DA - D = -U$ is the symmetric matrix with the elements

$$\begin{aligned} &d_{11} = a_2^2 d_{33}, \qquad d_{12} = \rho a_2^2 d_{33}, \qquad d_{13} = \rho a_2 d_{33}, \\ &d_{22} = a_2^2 d_{33}, \qquad d_{23} = \rho a_2 (a_0 + a_2) d_{33}, \\ &\rho = \frac{a_0}{1 - (a_0 + a_2) a_2}, \qquad d_{33} = \left(1 - a_2^2 - a_0^2 \frac{1 + (a_0 + a_2) a_2}{1 - (a_0 + a_2) a_2}\right)^{-1}. \end{aligned} \tag{8.18}$$

By the conditions

$$\begin{aligned} &|a_0| + (a_0 + a_2) a_2 < 1, \\ &a_2^2 + a_0^2 \frac{1 + (a_0 + a_2) a_2}{1 - (a_0 + a_2) a_2} < 1, \end{aligned} \tag{8.19}$$

the matrix D is a positively semidefinite matrix with $d_{33} > 0$ and the function $v_i = y'(i)Dy(i)$ is (via Sect. 5.1) a Lyapunov function for the auxiliary equation. In fact,

$$\begin{aligned}\Delta v_i &= y'(i+1)Dy(i+1) - y'(i)Dy(i) = y'(i)\big[A'DA - D\big]y(i)\\ &= -y'(i)Uy(i) = -y_i^2.\end{aligned}$$

Put now $x(i) = (x_{i-2}, x_{i-1}, x_i)'$, $\eta(i) = (0, 0, \eta_i)'$. Then (8.17) can be rewritten in the form $x(i+1) = \eta(i+1) + Ax(i)$. Putting $V_i = x'(i)Dx(i)$ and calculating $\mathbf{E}\Delta V_i$, we get

$$\begin{aligned}\mathbf{E}\Delta V_i &= \mathbf{E}\big[x'(i+1)Dx(i+1) - x'(i)Dx(i)\big]\\ &= -\mathbf{E}x'(i)Ux(i) + \mathbf{E}\eta'(i+1)D\eta(i+1) + 2\mathbf{E}\eta'(i+1)DAx(i)\\ &= -\mathbf{E}x_i^2 + d_{33}\mathbf{E}\eta_{i+1}^2 + 2(d_{23} + d_{33}a_0)\mathbf{E}x_i\eta_{i+1}\\ &\quad + 2d_{13}\mathbf{E}x_{i-1}\eta_{i+1} + 2d_{33}a_2\mathbf{E}x_{i-2}\eta_{i+1}.\end{aligned}$$

From (8.18) it follows that $d_{23} + d_{33}a_0 = \rho d_{33}$. Thus, for $\lambda > 0$ we have

$$\begin{aligned}\mathbf{E}\Delta V_i &= -\mathbf{E}x_i^2 + d_{33}\mathbf{E}\eta_{i+1}^2\\ &\qquad + 2d_{33}(\rho\mathbf{E}x_i\eta_{i+1} + \rho a_2\mathbf{E}x_{i-1}\eta_{i+1} + a_2\mathbf{E}x_{i-2}\eta_{i+1})\\ &\le -\mathbf{E}x_i^2 + \lambda d_{33}\big(|\rho|\mathbf{E}x_i^2 + |\rho a_2|\mathbf{E}x_{i-1}^2 + |a_2|\mathbf{E}x_{i-2}\big)\\ &\qquad + d_{33}\big[1 + \lambda^{-1}\big(|\rho| + |\rho a_2| + |a_2|\big)\big]\mathbf{E}\eta_{i+1}^2.\end{aligned}$$

There exists small enough $\lambda > 0$ such that $\lambda d_{33}(|\rho| + |\rho a_2| + |a_2|) < 1$. So, the functional V_i satisfies the conditions of Corollary 8.1. Therefore, if the conditions (8.19) hold and the sequence η_i, $i \in Z$, is mean square summable, then the solution of (8.17) is mean square summable. The corresponding region of summability is shown in Fig. 5.1 (number 1) by $a = a_0$, $b = a_2$.

8.4 Linear Equations with Variable Coefficients

Here the proposed procedure of the construction of Lyapunov functionals is applied to a linear Volterra difference equation with variable coefficients

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} a_{ij}x_j, \quad i \in Z,\ x_0 = \eta_0. \tag{8.20}$$

8.4.1 First Way of the Construction of the Lyapunov Functional

Represent the right-hand side of (8.20) in the form (1.7) by $\tau = 0$,

$$F_1(i, x_i) = a_{ii} x_i, \qquad F_2(i, x_0, \dots, x_i) = \sum_{l=0}^{i-1} a_{il} x_l, \qquad F_3(i, x_0, \dots, x_i) = 0.$$

Putting $V_i = x_i^2$ we obtain

$$\begin{aligned}\mathbf{E}\Delta V_i &= \mathbf{E}\big(x_{i+1}^2 - x_i^2\big) = -\mathbf{E}x_i^2 + \mathbf{E}\left(\eta_{i+1} + \sum_{j=0}^{i} a_{ij} x_j\right)^2 \\ &= -\mathbf{E}x_i^2 + \mathbf{E}\eta_{i+1}^2 + I_1 + I_2,\end{aligned}$$

where

$$I_1 = 2\mathbf{E}\eta_{i+1} \sum_{j=0}^{i} a_{ij} x_j, \qquad I_2 = \mathbf{E}\left(\sum_{j=0}^{i} a_{ij} x_j\right)^2.$$

Let $\lambda > 0$ and

$$\begin{aligned} \alpha_1 &= \sup_{i \in Z} \sum_{j=0}^{i} |a_{ij}|, \qquad \alpha_2 = \sup_{j \in Z} \sum_{i=j}^{\infty} |a_{ij}|, \\ A_{ij} &= |a_{ij}| \sum_{l=0}^{i} |a_{il}|, \quad A = \sup_{j \in Z} \sum_{i=j}^{\infty} A_{ij}. \end{aligned} \tag{8.21}$$

Then

$$|I_1| \le \sum_{j=0}^{i} |a_{ij}| \left(\frac{1}{\lambda}\mathbf{E}\eta_{i+1}^2 + \lambda \mathbf{E}x_j^2\right) \le \frac{\alpha_1}{\lambda}\mathbf{E}\eta_{i+1}^2 + \lambda \sum_{j=0}^{i} |a_{ij}| \mathbf{E}x_j^2,$$

$$I_2 \le \sum_{j=0}^{i} A_{ij} \mathbf{E}x_j^2,$$

and

$$\mathbf{E}\Delta V_i \le -\mathbf{E}x_i^2 + \sum_{j=0}^{i} \big(A_{ij} + \lambda |a_{ij}|\big)\mathbf{E}x_j^2 + \left(1 + \frac{\alpha_1}{\lambda}\right)\mathbf{E}\eta_{i+1}^2.$$

Note that

$$\sup_{i \in Z} \sum_{j=i}^{\infty} \big(A_{ji} + \lambda |a_{ji}|\big) \le A + \lambda \alpha_2.$$

So, if $A<1$ then for small enough $\lambda>0$ we have $A+\lambda\alpha_2<1$ and via Corollary 8.1 the solution of (8.20) is mean square summable for each mean square summable η_i, $i\in Z$.

Remark 8.2 Via (8.21)

$$A=\sup_{i\in Z}\sum_{j=i}^{\infty}|a_{ji}|\sum_{l=0}^{j}|a_{jl}|\le\alpha_1\alpha_2.$$

So, if $\alpha_1\alpha_2<1$ then $A<1$ and the solution of (8.20) is mean square summable.

Remark 8.3 In the stationary case, i.e., $a_{ij}=a_{i-j}$ we have $\alpha_1=\alpha_2=\sum_{j=0}^{\infty}|a_j|$, $A=\alpha_1^2$. So, the sufficient condition for the mean square summability of the solution of (8.20) takes the form $\alpha_1<1$, which coincides with (8.15).

8.4.2 Second Way of the Construction of the Lyapunov Functional

Represent the right-hand side of (8.20) in the form (1.7) with $\tau=0$,

$$F_1(i,x_i)=\beta_i x_i,\quad \beta_i=\sum_{j=i}^{\infty}a_{ji},$$

$$F_2(i,x_0,\dots,x_i)=0,\qquad F_3(i)=F_3(i,x_0,\dots,x_i)=-\sum_{j=0}^{i-1}x_j\sum_{l=i}^{\infty}a_{lj}.$$

By the condition

$$\sup_{i\in Z}|\beta_i|<1 \tag{8.22}$$

the solution of the auxiliary difference equation $y_{i+1}=\beta_i y_i$ is asymptotically stable.

Put $V_i=(x_i-F_3(i))^2$. Then via the representation $x_{i+1}-F_3(i+1)=\eta_{i+1}+\beta_i x_i-F_3(i)$ we have

$$\begin{aligned}
\mathbf{E}\Delta V_i&=\mathbf{E}\big[\big(x_{i+1}-F_3(i+1)\big)^2-\big(x_i-F_3(i)\big)^2\big]\\
&=\mathbf{E}\big(\eta_{i+1}+\beta_i x_i-F_3(i)-x_i+F_3(i)\big)\big(\eta_{i+1}+\beta_i x_i-F_3(i)+x_i-F_3(i)\big)\\
&=\mathbf{E}\big(\eta_{i+1}+(\beta_i-1)x_i\big)\big(\eta_{i+1}+(\beta_i+1)x_i-2F_3(i)\big)\\
&=\mathbf{E}\eta_{i+1}^2+\big(\beta_i^2-1\big)\mathbf{E}x_i^2+I_1+I_2+I_3,
\end{aligned}$$

where

$$I_1=2\beta_i\mathbf{E}\eta_{i+1}x_i,\qquad I_2=-2\mathbf{E}\eta_{i+1}F_3(i),\qquad I_3=2(1-\beta_i)\mathbf{E}x_iF_3(i).$$

Suppose also that

$$B_{ij}=\left|\sum_{l=i}^{\infty}a_{lj}\right|,\quad \alpha=\sup_{i\geq 1}\sum_{j=0}^{i-1}B_{ij}<\infty,\quad \sup_{j\in Z}\sum_{i=j+1}^{\infty}B_{ij}<\infty. \tag{8.23}$$

Then via (8.22), (8.23) and $\lambda>0$

$$|I_1|\leq \frac{1}{\lambda}\mathbf{E}\eta_{i+1}+\lambda\mathbf{E}x_i^2,$$

$$|I_2|\leq \sum_{j=0}^{i-1}B_{ij}\left(\frac{1}{\lambda}\mathbf{E}\eta_{i+1}^2+\lambda\mathbf{E}x_j^2\right)\leq \frac{\alpha}{\lambda}\mathbf{E}\eta_{i+1}^2+\lambda\sum_{j=0}^{i-1}B_{ij}\mathbf{E}x_j^2,$$

$$|I_3|\leq (1-\beta_i)\sum_{j=0}^{i-1}B_{ij}\big(\mathbf{E}x_i^2+\mathbf{E}x_j^2\big)=(1-\beta_i)\left(\mathbf{E}x_i^2\sum_{j=0}^{i-1}B_{ij}+\sum_{j=0}^{i-1}B_{ij}\mathbf{E}x_j^2\right).$$

As a result we obtain

$$\begin{aligned}\mathbf{E}\Delta V_i&\leq \mathbf{E}\eta_{i+1}^2+\big(\beta_i^2-1\big)\mathbf{E}x_i^2+\frac{1}{\lambda}\mathbf{E}\eta_{i+1}+\lambda\mathbf{E}x_i^2\\&\quad+\frac{\alpha}{\lambda}\mathbf{E}\eta_{i+1}^2+\lambda\sum_{j=0}^{i-1}B_{ij}\mathbf{E}x_j^2+(1-\beta_i)\left(\mathbf{E}x_i^2\sum_{j=0}^{i-1}B_{ij}+\sum_{j=0}^{i-1}B_{ij}\mathbf{E}x_j^2\right)\\&=-(1-\lambda)\mathbf{E}x_i^2+\sum_{j=0}^{i}A_{ij}\mathbf{E}x_j^2+\left(1+\frac{1+\alpha}{\lambda}\right)\mathbf{E}\eta_{i+1}^2,\end{aligned}$$

where

$$A_{ij}=\begin{cases}(1-\beta_i+\lambda)B_{ij}, & j<i,\\ \beta_i^2+(1-\beta_i)\sum_{l=0}^{i-1}B_{il}, & j=i.\end{cases}$$

So, if

$$\sup_{j\in Z}\left(\beta_j^2+(1-\beta_j)\sum_{i=0}^{j-1}B_{ji}+\sum_{i=j+1}^{\infty}(1-\beta_i)B_{ij}\right)<1 \tag{8.24}$$

then via (8.23) for small enough $\lambda>0$ the functional V_i satisfies the conditions of Corollary 1.1 and therefore by the conditions (8.22)–(8.24) the solution of (8.20) is mean square summable for each mean square summable η_i, $i\in Z$.

Remark 8.4 In the stationary case, i.e. $a_{ij} = a_{i-j}$, the condition (8.24) coincides with (8.16). In fact, in this case $\beta_i = \beta$,

$$\sum_{i=j+1}^{\infty} B_{ij} = \sum_{i=j+1}^{\infty} \left| \sum_{l=i}^{\infty} a_{l-j} \right| = \sum_{m=1}^{\infty} \left| \sum_{l=m}^{\infty} a_l \right| = \alpha,$$

$$\sup_{j\geq 1} \sum_{i=0}^{j-1} \left| \sum_{l=j}^{\infty} a_{l-i} \right| = \sup_{j\geq 1} \sum_{i=0}^{j-1} \left| \sum_{m=j-i}^{\infty} a_m \right| = \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} a_m \right| = \alpha.$$

8.4.3 Resolvent Representation

In this item stability conditions for the solution of the difference equation (8.20) are obtained by virtue of the resolvent b_{ij} of the kernel a_{ij}.

Lemma 8.1 *The solution of difference equation* (8.20) *can be represented in the form*

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} b_{ij} \eta_j, \quad x_0 = \eta_0,\ i \in Z, \tag{8.25}$$

where the numbers b_{ij} are defined by recurrent formulas

$$b_{ij} = a_{ij} + \sum_{l=j+1}^{i} a_{il} b_{l-1,j}, \quad 0 \leq j \leq i, \tag{8.26}$$

or

$$b_{ij} = a_{ij} + \sum_{l=j+1}^{i} b_{il} a_{l-1,j}, \quad 0 \leq j \leq i. \tag{8.27}$$

Proof Substituting (8.25) into (8.20) we obtain

$$\eta_{i+1} + \sum_{j=0}^{i} b_{ij} \eta_j = \eta_{i+1} + \sum_{j=0}^{i} a_{ij} \left(\eta_j + \sum_{l=0}^{j-1} b_{j-1,l} \eta_l \right).$$

From this for an arbitrary sequence η_j, $j \in Z$, it follows that

$$\sum_{j=0}^{i} b_{ij} \eta_j = \sum_{j=0}^{i} a_{ij} \eta_j + \sum_{j=0}^{i} \left(\sum_{l=j+1}^{i} a_{il} b_{l-1,j} \right) \eta_j$$

or (8.26). Substituting (8.20) into (8.25) we obtain

$$x_{i+1} = x_{i+1} - \sum_{j=0}^{i} a_{ij} x_j + \sum_{j=0}^{i} b_{ij} \left(x_j - \sum_{l=0}^{j-1} a_{j-1,l} x_l \right).$$

From this, for an arbitrary sequence x_j, $j \in Z$, it follows that

$$\sum_{j=0}^{i} b_{ij} x_j = \sum_{j=0}^{i} a_{ij} x_j + \sum_{l=0}^{i} \left(\sum_{l=j+1}^{i} b_{il} a_{l-1,j} \right) x_j$$

or (8.27). The lemma is proven. □

Remark 8.5 In stationary case, i.e. $a_{ij} = a_{i-j}$ and $b_{ij} = b_{i-j}$, from (8.26) it follows that

$$b_i = a_i + \sum_{l=1}^{i} a_{i-l} b_{l-1}, \quad i = 0, 1, \ldots .$$

In particular,

$$b_0 = a_0, \quad b_1 = a_1 + a_0 b_0 = a_1 + a_0^2,$$
$$b_2 = a_2 + a_1 b_0 + a_0 b_1 = a_2 + 2a_1 a_0 + a_0^3, \quad \ldots .$$

Theorem 8.2 *If*

$$\beta_1 = \sup_{i \in Z} \sum_{j=0}^{i} |b_{ij}| < \infty \tag{8.28}$$

then the solution of (8.20) *is mean square stable. If besides*

$$\beta_2 = \sup_{j \in Z} \sum_{i=j}^{\infty} |b_{ij}| < \infty \tag{8.29}$$

then the solution of (8.20) *is mean square summable for each mean square summable* $\eta_i, i \in Z$.

Proof From (8.25) we get

$$\mathbf{E}|x_{i+1}|^2 \le 2 \left(\mathbf{E}|\eta_{i+1}|^2 + \sum_{l=0}^{i} |b_{il}| \sum_{j=0}^{i} |b_{ij}| \mathbf{E}|\eta_j|^2 \right). \tag{8.30}$$

Let $\varepsilon > 0$. If $\|\eta\|^2 < \delta = \frac{\varepsilon}{2(1+\beta_1^2)}$ then via (8.28) $\mathbf{E}|x_{i+1}|^2 \le 2(1+\beta_1^2)\delta = \varepsilon$. Thus, the solution of (8.20) is mean square stable.

Via (8.30) and (8.29) we have

$$\sum_{i=0}^{\infty}\mathbf{E}|x_{i+1}|^2 \le 2\left(\sum_{i=0}^{\infty}\mathbf{E}|\eta_{i+1}|^2 + \sum_{i=0}^{\infty}\sum_{l=0}^{i}|b_{il}|\sum_{j=0}^{i}|b_{ij}|\mathbf{E}|\eta_j|^2\right)$$
$$\le 2\left(\sum_{i=0}^{\infty}\mathbf{E}|\eta_{i+1}|^2 + \beta_1\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}|b_{ij}|\mathbf{E}|\eta_j|^2\right)$$
$$\le 2(1+\beta_1\beta_2)\sum_{j=0}^{\infty}\mathbf{E}|\eta_j|^2.$$

Thus, if the sequence η_i is mean square summable then the solution of (8.20) is mean square summable too. The theorem is proven. □

Theorem 8.3 *If*

$$\alpha_1 = \sup_{i\in Z}\sum_{j=0}^{i}|a_{ij}| < 1 \tag{8.31}$$

then the solution of (8.20) *is mean square stable. If besides*

$$\alpha_2 = \sup_{j\in Z}\sum_{i=j}^{\infty}|a_{ij}| < 1 \tag{8.32}$$

then the solution of (8.20) *is mean square summable for each mean square summable* $\eta_i, i \in Z$.

Proof From (8.27) we get

$$\sum_{j=0}^{i}|b_{ij}| \le \sum_{j=0}^{i}|a_{ij}| + \sum_{j=0}^{i}\sum_{l=j+1}^{i}|b_{il}||a_{l-1,j}|$$
$$= \sum_{j=0}^{i}|a_{ij}| + \sum_{l=1}^{i}|b_{il}|\sum_{j=0}^{l-1}|a_{l-1,j}| \le \alpha_1 + \alpha_1\sum_{l=0}^{i}|b_{il}|.$$

From this and (8.31) we obtain $B \le \alpha_1(1-\alpha_1)^{-1}$, i.e., (8.28). Via Theorem 8.2 it means that the solution of (8.20) is mean square stable.

Via (8.26) and (8.32) we have

$$\sum_{i=j}^{\infty}|b_{ij}| \le \sum_{i=j}^{\infty}|a_{ij}| + \sum_{i=j}^{\infty}\sum_{l=j+1}^{i}|a_{il}b_{l-1,j}|$$
$$= \sum_{i=j}^{\infty}|a_{ij}| + \sum_{l=j+1}^{\infty}|b_{l-1,j}|\sum_{i=l}^{\infty}|a_{il}| \le \alpha_2 + \alpha_1\sum_{i=j}^{\infty}|b_{ij}|.$$

From this and (8.32) follows (8.29) and therefore the solution of (8.20) is mean square summable for each mean square summable $\eta_i, i \in Z$. The theorem is proven. □

Remark 8.6 In stationary case, i.e. $a_{ij} = a_{i-j}$ and $b_{ij} = b_{i-j}$, the conditions (8.28), (8.29), (8.31) and (8.32) take the form

$$\alpha_1 = \alpha_2 = \sum_{j=0}^{\infty} |a_j| < 1, \qquad \beta_1 = \beta_2 = \sum_{j=0}^{\infty} |b_j| < \infty.$$

As follows from Theorems 8.2 and 8.3, in this case the solution of (8.20) is mean square stable, mean square summable and asymptotically mean square trivial for each mean square summable $\eta_i, i \in Z$.

Consider now the equation with degenerate kernel

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} p_i' q_j x_j, \quad x_0 = \eta_0, \ i \in Z. \tag{8.33}$$

Here x_i and η_i are scalars, $p_i' = (p_1^{(i)}, \ldots, p_n^{(i)})$, $q_j' = (q_1^{(j)}, \ldots, q_n^{(j)})$.

Corollary 8.2 *Put $G_l = q_l p_{l-1}'$. The inequalities*

$$\begin{aligned} &\sup_{i \geq 0} \sum_{j=0}^{i} \left| p_i' \left(\prod_{k=0}^{i-j-1} (I + G_{i-k}) \right) q_j \right| < \infty, \\ &\sup_{j \in Z} \sum_{i=j}^{\infty} \left| p_i' \left(\prod_{k=0}^{i-j-1} (I + G_{i-k}) \right) q_j \right| < \infty, \end{aligned} \tag{8.34}$$

or the inequalities

$$\sup_{i \in Z} \sum_{j=0}^{i} |p_i' q_j| < 1, \qquad \sup_{j \in Z} \sum_{i=j}^{\infty} |p_i' q_j| < 1 \tag{8.35}$$

are sufficient conditions for the mean square stability and mean square summability of the of solution equation (8.33).

Proof Let b_{ij} be the resolvent of the kernel $p_i' q_j$. Represent b_{ij} in the form $b_{ij} = p_i' P_{ij} q_j$ where P_{ij} is an unknown matrix. Via (8.26) and (8.27), we have

$$p_i' P_{ij} q_j = p_i' q_j + \sum_{l=j+1}^{i} p_i' G_l P_{l-1,j} q_j, \qquad p_i' P_{ij} q_j = p_i' q_j + \sum_{l=j+1}^{i} p_i' P_{il} G_l q_j.$$

From this it follows that $P_{ii} = I$,

$$P_{ij} = I + \sum_{l=j+1}^{i} G_l P_{l-1,j}, \qquad P_{ij} = I + \sum_{l=j+1}^{i} P_{il} G_l, \quad i > j,$$

where I is $n \times n$-identity matrix. Then

$$P_{ij} = I + \sum_{l=j+1}^{i} G_l P_{l-1,j} = P_{i-1,j} + G_i P_{i-1,j} = (I + G_i) P_{i-1,j},$$

$$P_{ij} = I + \sum_{l=j+2}^{i} P_{il} G_l + P_{i,j+1} G_{j+1} = P_{i,j+1}(I + G_{j+1}).$$

Elementary calculations show that

$$P_{ij} = \prod_{k=0}^{i-j-1} (I + G_{i-k}), \quad i > j.$$

Therefore, the resolvent b_{ij} has the representation

$$b_{ij} = p_i' \prod_{k=0}^{i-j-1} (I + G_{i-k}) q_j, \quad i > j.$$

Via Theorems 8.2 and 8.3 the corollary is proven. □

Note that the conditions (8.34) and (8.35) are defined immediately in terms of the parameters of (8.33).

Example 8.2 Consider difference equation (8.20) with the kernel

$$a_{ij} = \lambda \frac{(j+1)^\gamma}{(i+2)^{\gamma+1}}, \quad 0 \le j \le i, \ \gamma > 0. \tag{8.36}$$

Using Lemma 1.4 for estimating α_1 and α_2 we obtain

$$\alpha_1 = \sup_{i \in Z} \sum_{j=0}^{i} |a_{ij}| = \sup_{i \in Z} \frac{|\lambda|}{(i+2)^{\gamma+1}} \sum_{j=1}^{i+1} j^\gamma$$

$$\le \sup_{i \in Z} \frac{|\lambda|}{(i+2)^{\gamma+1}} \int_1^{i+2} x^\gamma \, dx \le \frac{|\lambda|}{\gamma+1},$$

$$\alpha_2 = \sup_{j\in Z}\sum_{i=j}^{\infty}|a_{ij}| = \sup_{j\in Z}|\lambda|(j+1)^{\gamma}\sum_{i=j+2}^{\infty}\frac{1}{i^{\gamma+1}}$$
$$\le \sup_{j\in Z}|\lambda|(j+1)^{\gamma}\int_{j+1}^{\infty}\frac{\mathrm{d}x}{x^{\gamma+1}} = \frac{|\lambda|}{\gamma}.$$

So, via Theorem 8.3, if $|\lambda| < \gamma + 1$ then the solution of (8.20) is mean square stable, if $|\lambda| < \gamma$ then the solution of (8.20) is mean square stable and mean square summable for each mean square summable η_i, $i \in Z$.

Note that Remark 8.2 gives another sufficient condition for the mean square summability:

$$|\lambda| < \sqrt{\gamma(\gamma+1)}, \tag{8.37}$$

which is weaker than $|\lambda| < \gamma$.

Example 8.3 Consider the difference equation (8.20) with the kernel

$$a_{ij} = \frac{(j+1)^{\gamma}}{(i+2)^{\gamma+1}}, \quad 0 \le j \le i,\ \gamma > 1. \tag{8.38}$$

Kernel (8.38) is a degenerate one; therefore, (8.20) with kernel (8.38) can be considered as (8.33) with

$$p_i = \frac{1}{(i+2)^{\gamma+1}}, \qquad q_j = (j+1)^{\gamma}. \tag{8.39}$$

As follows from Example 8.2, if $|\lambda| < \gamma$ then the conditions (8.35) hold, so, via Corollary 8.2 the solution of (8.33) and (8.39) is mean square stable and mean square summable.

Let us show that the conditions (8.34) hold too. In fact, via (8.39) $G_l = (l+1)^{-1}$. Therefore,

$$\prod_{k=0}^{i-j-1}(1+G_{i-k}) = \left(1+\frac{1}{i+1}\right)\left(1+\frac{1}{i}\right)\cdots\left(1+\frac{1}{j+2}\right) = \frac{i+2}{j+2},$$

and via Lemma 1.4 we obtain

$$\sum_{j=0}^{i}\frac{1}{j+2}\left(\frac{j+1}{i+2}\right)^{\gamma} \le \frac{1}{(i+2)^{\gamma}}\sum_{j=1}^{i+1}j^{\gamma-1}$$
$$\le \frac{1}{(i+2)^{\gamma}}\int_{1}^{i+2}x^{\gamma-1}\,\mathrm{d}x \le \frac{1}{\gamma(i+2)^{\gamma-1}} \le \frac{1}{\gamma 2^{\gamma-1}},$$

$$\sum_{i=j}^{\infty} \frac{1}{j+2}\left(\frac{j+1}{i+2}\right)^{\gamma} \leq (j+1)^{\gamma-1} \sum_{i=j+2}^{\infty} \frac{1}{j^{\gamma}}$$
$$\leq (j+1)^{\gamma-1} \int_{j+1}^{\infty} \frac{\mathrm{d}x}{x^{\gamma}} \leq \frac{1}{\gamma-1}.$$

Example 8.4 Consider the difference equation (8.20) with the kernel $a_{ij} = \lambda a^i b^j$ that satisfies the conditions

$$|a| + |\lambda| < 1, \qquad |ab| \leq 1. \tag{8.40}$$

Let us show that by the conditions (8.40), the conditions (8.35) hold. In fact,

$$\sum_{j=0}^{i} \left|\lambda a^i b^j\right| \leq |\lambda| \sum_{j=0}^{i} |a|^{i-j} |ab|^j \leq \frac{|\lambda|}{1-|a|} < 1,$$
$$\sum_{i=j}^{\infty} \left|\lambda a^i b^j\right| \leq |\lambda| \sum_{i=j}^{\infty} |a|^{i-j} |ab|^j \leq \frac{|\lambda|}{1-|a|} < 1.$$

The conditions (8.34) hold too. In fact, $G_l = \lambda a^{l-1} b^l = \lambda a^{-1}(ab)^l$. Thus,

$$\sum_{j=0}^{i} \left| p_i' \left(\prod_{k=0}^{i-j-1} (1 + G_{i-k}) \right) q_j \right|$$
$$= |\lambda| \sum_{j=0}^{i} \left| a^{i-j} \left(\prod_{k=0}^{i-j-1} \left(1 + \frac{\lambda}{a}(ab)^{i-k} \right) \right) (ab)^j \right|$$
$$\leq |\lambda| \sum_{j=0}^{i} \left| a^{i-j} \left(\prod_{k=0}^{i-j-1} \left(1 + \frac{|\lambda|}{|a|} \right) \right) \right| = |\lambda| \sum_{j=0}^{i} |a|^{i-j} \left(1 + \frac{|\lambda|}{|a|} \right)^{i-j}$$
$$= |\lambda| \sum_{j=0}^{i} \left(|a| + |\lambda|\right)^{i-j} \leq \frac{|\lambda|}{1 - (|a| + |\lambda|)} < \infty$$

and similar for the second inequality.

8.5 Nonlinear Systems

Here the procedure of the construction of Lyapunov functionals is used for some nonlinear Volterra equations.

8.5.1 Stationary Systems

Consider the nonlinear difference equation

$$x_{i+1} = \eta_{i+1} + Ax_i + \sum_{j=0}^{i} F(i-j, x_j). \tag{8.41}$$

Here $x_i, \eta_i, F(i,x) \in R^n$, $\eta = \{\eta_i, i \in Z\} \in H_2$, A is a $n \times n$-matrix, $|F(i,x)| \le a_i|x|$,

$$\beta = \sum_{j=0}^{\infty} a_j < \infty. \tag{8.42}$$

Theorem 8.4 *Let D be a positive solution of the matrix equation $A'DA - D = -I$ (here I is the identity $n \times n$-matrix) and let it satisfy the condition*

$$\beta^2\|D\| + 2\beta\|DA\| < 1. \tag{8.43}$$

Then the solution of (8.41) is mean square summable for each mean square summable $\eta_i, i \in Z$.

Proof Put $V_i = x_i' D x_i$, Calculating $\mathbf{E}\Delta V_i$, we get

$$\begin{aligned}\mathbf{E}\Delta V_i &= \mathbf{E}\Bigg[\Bigg(\eta_{i+1} + Ax_i + \sum_{j=0}^{i} F(i-j, x_j)\Bigg)' D \\ &\quad \times \Bigg(\eta_{i+1} + Ax_i + \sum_{j=0}^{i} F(i-j, x_j)\Bigg) - x_i' D x_i\Bigg] \\ &\le \|D\|\mathbf{E}|\eta_{i+1}|^2 - \mathbf{E}|x_i|^2 + \sum_{l=1}^{4} I_l,\end{aligned}$$

where

$$I_1 = 2\mathbf{E}\eta_{i+1}' DAx_i, \qquad I_2 = 2\sum_{j=0}^{i} \mathbf{E}\eta_{i+1}' DF(i-j, x_j),$$

$$I_3 = 2\sum_{j=0}^{i} \mathbf{E}F'(i-j, x_j) DAx_i, \qquad I_4 = \sum_{j=0}^{i}\sum_{l=0}^{i} \mathbf{E}F'(i-j, x_j) DF(i-l, x_l).$$

Via (8.42) and some $\lambda > 0$ we have

$$|I_1| \le \|DA\|\left(\frac{1}{\lambda}\mathbf{E}|\eta_{i+1}|^2 + \lambda\mathbf{E}|x_i|^2\right),$$

$$
\begin{aligned}
|I_2| &\le \|D\| \sum_{j=0}^{i} a_{i-j} \left(\frac{1}{\lambda} \mathbf{E}|\eta_{i+1}|^2 + \lambda \mathbf{E}|x_j|^2 \right) \\
&\le \|D\| \left(\frac{\beta}{\lambda} \mathbf{E}|\eta_{i+1}|^2 + \lambda a_0 \mathbf{E}|x_i|^2 + \lambda \sum_{j=0}^{i-1} a_{i-j} \mathbf{E}|x_j|^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\le \|DA\| \sum_{j=0}^{i} a_{i-j} \big(\mathbf{E}|x_i|^2 + \mathbf{E}|x_j|^2 \big) \\
&\le \|DA\| \left((\beta + a_0) \mathbf{E}|x_i|^2 + \sum_{j=0}^{i-1} a_{i-j} \mathbf{E}|x_j|^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
I_4 &\le \|D\| \mathbf{E} \left(\sum_{j=0}^{i} \big| F(i-j, x_j) \big| \right)^2 \le \|D\| \mathbf{E} \left(\sum_{j=0}^{i} a_{i-j} |x_j| \right)^2 \\
&\le \|D\| \sum_{j=0}^{i} a_{i-j} \sum_{j=0}^{i} a_{i-j} \mathbf{E}|x_j|^2 \le \beta \|D\| \sum_{j=0}^{i} a_{i-j} \mathbf{E}|x_j|^2 \\
&= \beta \|D\| \left(a_0 \mathbf{E}|x_i|^2 + \sum_{j=0}^{i-1} a_{i-j} \mathbf{E}|x_j|^2 \right).
\end{aligned}
$$

As a result we obtain

$$
\mathbf{E}\Delta V_i \le -c\mathbf{E}|x_i|^2 + \sum_{j=0}^{i-1} A_{ij} \mathbf{E}|x_j|^2 + \left(\|D\| + \frac{\beta\|D\| + \|DA\|}{\lambda} \right) \mathbf{E}|\eta_{i+1}|^2,
$$

where

$$
\begin{aligned}
c &= 1 - \beta a_0 \|D\| - (\beta + a_0)\|DA\| - \lambda\big(a_0\|D\| + \|DA\|\big), \\
A_{i-j} &= \big(\beta\|D\| + \|DA\| + \lambda\|D\|\big) a_{i-j}, \quad i > j.
\end{aligned}
$$

If the condition (8.43) holds then there exists a small enough $\lambda > 0$ such that

$$
\beta^2 \|D\| + 2\beta\|DA\| + \lambda\big(\beta\|D\| + \|DA\|\big) < 1
$$

and therefore the functional V_i satisfies Corollary 1.1. Thus, if the condition (8.43) holds, then the solution of (8.41) is mean square summable for each mean square summable sequence η_i. The proof is completed. □

Remark 8.7 Put in (8.41) $A = 0$. Then $D = I$ and the condition (8.43) takes the form $\alpha_1 < 1$.

Remark 8.8 If (8.41) is a scalar one then $D = (1 - A^2)^{-1}$ and the condition (8.43) takes the form $\beta + |A| < 1$.

Consider now the system of two scalar difference equations

$$x_{i+1} = \eta_{i+1} + \sum_{l=0}^{i} a_{i-l} x_l, \qquad y_{i+1} = c y_i \left(1 + \sum_{j=0}^{i} b_j x_j^2\right)^{-1}. \tag{8.44}$$

Here $b_j \geq 0$, a_j and c are known constants. Put

$$\alpha_1 = \sum_{l=0}^{\infty} |a_l| < \infty.$$

Using the functional $V_i = x_i^2 + y_i^2$ and $\lambda > 0$, we have

$$\begin{aligned}
\mathbf{E}\Delta V_i &= \mathbf{E}\big(x_{i+1}^2 + y_{i+1}^2 - x_i^2 - y_i^2\big) \\
&= \mathbf{E}\left[-x_i^2 + \left(\eta_{i+1} + \sum_{l=0}^{i} a_{i-l} x_l\right)^2 + \left(\frac{c^2}{(1+\sum_{j=0}^{i} b_j x_j^2)^2} - 1\right) y_i^2\right] \\
&\leq \mathbf{E}\left[-x_i^2 + \eta_{i+1}^2 + 2\sum_{l=0}^{i} a_{i-l}\eta_{i+1} x_l + \left(\sum_{l=0}^{i} a_{i-l} x_l\right)^2 + \big(c^2 - 1\big) y_i^2\right] \\
&\leq \mathbf{E}\left[-x_i^2 - \big(1 - c^2\big) y_i^2 + \eta_{i+1}^2 + \sum_{l=0}^{i} |a_{i-l}| \left(\frac{1}{\lambda}\eta_{i+1}^2 + \lambda x_l^2\right)\right. \\
&\qquad \left. + \alpha_1 \sum_{l=0}^{i} |a_{i-l}| x_l^2\right] \\
&\leq -\mathbf{E}x_i^2 - \big(1 - c^2\big)\mathbf{E}y_i^2 + (\alpha_1 + \lambda)\sum_{l=0}^{i} |a_{i-l}| x_l^2 + \big(1 + \lambda^{-1}\alpha_1\big)\mathbf{E}\eta_{i+1}^2.
\end{aligned}$$

From Corollary 8.1 it follows that if the conditions $|c| < 1$ and $|\alpha_1| < 1$ hold, then the solution of system (8.44) is mean square summable for each mean square summable sequence η_i.

8.5.2 *Nonstationary Systems*

Consider the nonstationary nonlinear Volterra difference equation

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} a_{ij} g(x_j), \quad x_0 = \eta_0. \tag{8.45}$$

Theorem 8.5 *Let the sequence η_i be mean summable, and let the kernel a_{ij} and the function $g(x)$ satisfy the conditions*

$$\alpha_2 = \sup_{j \in Z} \sum_{i=j}^{\infty} |a_{ij}| < 1, \qquad \left|g(x)\right| \le |x|. \tag{8.46}$$

Then the solution of (8.45) is mean summable.

Proof As follows from Theorem 8.1 it is enough to show that some functional $V_i = V(i, x_0, \ldots, x_i)$, $i \in Z$, satisfies the conditions (8.3) and (8.4) by $p = 1$. Put

$$V_0 = |x_0|, \quad V_i = \sum_{j=0}^{i-1} \sum_{l=i-1}^{\infty} \left|a_{lj} g(x_j)\right|, \quad i > 0.$$

This functional is bounded for each $i \ge 0$. In fact, from (8.46) it follows that

$$V_i \le \alpha_2 \sum_{j=0}^{i-1} \left|g(x_j)\right| < \infty, \quad i > 0.$$

For $i = 0$ via (8.46) we have

$$\Delta V_0 = \sum_{l=0}^{\infty} \left|a_{l0} g(x_0)\right| - |x_0| \le \alpha_2 \left|g(x_0)\right| - |x_0| \le -(1 - \alpha_2)|x_0|.$$

From (8.45) it follows that $-\sum_{j=0}^{i-1} |a_{i-1,j} g(x_j)| \le |\eta_i| - |x_i|$. So, via (8.46) for $i > 0$ we obtain

$$\begin{aligned} \Delta V_i &= \sum_{j=0}^{i} \sum_{l=i}^{\infty} \left|a_{lj} g(x_j)\right| - V_i = \left|g(x_i)\right| \sum_{l=i}^{\infty} |a_{li}| - \sum_{j=0}^{i-1} \left|a_{i-1,j} g(x_j)\right| \\ &\le \alpha_2 |x_i| + |\eta_i| - |x_i| = -(1 - \alpha_2)|x_i| + |\eta_i|. \end{aligned}$$

As a result, $\mathbf{E}\Delta V_i \le -(1 - \alpha_2)\mathbf{E}|x_i| + \mathbf{E}|\eta_i|$, i.e. conditions (8.4) by $p = 1$ hold. The proof is completed. □

Let us obtain a sufficient condition for p-summability of the solution of (8.45) by $p \ge 1$.

Theorem 8.6 *Let the sequence η_i be p-summable and the kernel a_{ij} satisfy the condition*

$$\alpha_2 \alpha_1^{p-1} < 1, \tag{8.47}$$

where α_2 *is defined by* (8.46) *and*

$$\alpha_1 = \sup_{i \in Z} \sum_{l=0}^{i} |a_{il}|. \tag{8.48}$$

Then the solution of (8.45) *is* p*-summable.*

Proof Let us show that the functional $V_i = V(i, x_0, \dots, x_i)$, $i \in Z$, where

$$V_0 = |x_0|^p, \quad V_i = \sum_{j=0}^{i-1} \sum_{l=i-1}^{\infty} |a_{lj}| \big|g(x_j)\big|^p, \quad i > 0,$$

satisfies the conditions (8.4). Note that the functional V_i is bounded for each $i \geq 0$ since from (8.46) it follows that

$$V_i \leq \alpha_2 \sum_{j=0}^{i-1} \big|g(x_j)\big|^p < \infty, \quad i > 0.$$

It is easy to see that

$$\Delta V_0 = \sum_{l=0}^{\infty} |a_{l0}| \big|g(x_0)\big|^p - |x_0|^p \leq -(1 - \alpha_2)|x_0|^p.$$

Calculating ΔV_i by $i > 0$ we have

$$\begin{aligned} \Delta V_i &= \sum_{j=0}^{i} \sum_{l=i}^{\infty} |a_{lj}| \big|g(x_j)\big|^p - V_i \\ &= \big|g(x_i)\big|^p \sum_{l=i}^{\infty} |a_{li}| - \sum_{j=0}^{i-1} |a_{i-1,j}| \big|g(x_j)\big|^p. \end{aligned} \tag{8.49}$$

Via Lemma 1.2 from (8.43) for some $\lambda > 0$ it follows that

$$\begin{aligned} |x_i|^p &\leq \left(|\eta_i| + \sum_{j=0}^{i-1} |a_{i-1,j}| \big|g(x_j)\big| \right)^p \\ &\leq (1 + \lambda)^{p-1} |\eta_i|^p + \left(1 + \frac{1}{\lambda}\right)^{p-1} \left(\sum_{j=0}^{i-1} |a_{i-1,j}| \big|g(x_j)\big| \right)^p. \end{aligned}$$

Via the Hölder inequality we have

$$
\begin{aligned}
&\left(\sum_{j=0}^{i-1}|a_{i-1,j}|\big|g(x_j)\big|\right)^p \\
&= \left(\sum_{j=0}^{i-1}|a_{i-1,j}|^{1-\frac{1}{p}}|a_{i-1,j}|^{\frac{1}{p}}\big|g(x_j)\big|\right)^p \\
&\le \left(\sum_{j=0}^{i-1}|a_{i-1,j}|\right)^{p-1}\sum_{j=0}^{i-1}|a_{i-1,j}|\big|g(x_j)\big|^p \le \alpha_1^{p-1}\sum_{j=0}^{i-1}|a_{i-1,j}|\big|g(x_j)\big|^p.
\end{aligned}
$$

Therefore,

$$
|x_i|^p \le (1+\lambda)^{p-1}|\eta_i|^p + \left(\frac{\alpha_1(1+\lambda)}{\lambda}\right)^{p-1}\sum_{j=0}^{i-1}|a_{i-1,j}|\big|g(x_j)\big|^p.
$$

From this it follows that

$$
-\sum_{j=0}^{i-1}|a_{i-1,j}|\big|g(x_j)\big|^p \le \left(\frac{\lambda}{\alpha_1}\right)^{p-1}|\eta_i|^p - \left(\frac{\lambda}{\alpha_1(1+\lambda)}\right)^{p-1}|x_i|^p.
$$

So, using (8.49) and (8.46) we obtain $\mathbf{E}\Delta V_i \le -c\mathbf{E}|x_i|^p + \gamma_i$, where

$$
c = \left(\frac{\lambda}{\alpha_1(1+\lambda)}\right)^{p-1} - \alpha_2, \qquad \gamma_i = \left(\frac{\lambda}{\alpha_1}\right)^{p-1}\mathbf{E}|\eta_i|^p.
$$

From (8.47) it follows that there exists a large enough $\lambda > 0$ such that

$$
\alpha_2\alpha_1^{p-1} < \left(\frac{\lambda}{1+\lambda}\right)^{p-1}
$$

or $c > 0$. Therefore via Corollary 8.1 the solution of (8.45) is p-summable. The proof is completed. □

Remark 8.9 If in (8.45) $a_{ij} = a_{i-j}$ then $\alpha_1 = \alpha_2 = \sum_{j=0}^{\infty}|a_j|$ and the inequality $\alpha_1 < 1$ is a sufficient condition for p-summability of the solution of (8.45) by $p \ge 1$. Let, for example, $a_i = \lambda q^i$, $i \in Z$, $|q| < 1$. Then $\alpha_1 = |\lambda|(1-|q|)^{-1}$ and the inequality $|\lambda| + |q| < 1$ is a sufficient condition for p-summability, $p \ge 1$.

Remark 8.10 If the kernel a_{ij} in (8.45) is degenerate, i.e. $a_{ij} = p_i q_j$, then

$$
\alpha_1 = \sup_{i\in Z}\left(|p_i|\sum_{j=0}^{i}|q_j|\right), \qquad \alpha_2 = \sup_{j\in Z}\left(|q_j|\sum_{i=j}^{\infty}|p_i|\right).
$$

8.5.3 Nonstationary System with Monotone Coefficients

In some cases for some systems of special type it is possible to get stability conditions using special characteristics of parameters of the system under consideration without too restrictive conditions type of (8.47). Consider the nonlinear difference equation

$$x_{i+1} = \eta_{i+1} - \sum_{j=0}^{i} a_{ij} f(x_j), \tag{8.50}$$

where the function $f(x)$ satisfies the condition

$$0 < c_1 \le \frac{f(x)}{x} \le c_2, \quad x \neq 0. \tag{8.51}$$

Theorem 8.7 *Let the coefficients a_{ij}, $i \in Z$, $j = 0, \dots, i$, satisfy the conditions*

$$a_{ij} \ge a_{i,j-1} \ge 0, \tag{8.52}$$

$$a_{i+1,j-1} - a_{i+1,j} - a_{i,j-1} + a_{ij} \ge 0, \tag{8.53}$$

$$a = \sup_{i \in Z}(a_{i+1,i+1} + a_{ii} - a_{i+1,i}) < \frac{2}{c_2}, \tag{8.54}$$

$$\alpha_1 = \sup_{i \in Z} \sum_{j=0}^{i} a_{ij} < \infty, \qquad \alpha_2 = \sup_{j \in Z} \sum_{i=j}^{\infty} a_{ij} < \infty \tag{8.55}$$

(*here it is supposed that* $a_{i,-1} = 0$). *Then the solution of* (8.50) *is mean square summable.*

Proof Let us construct the functional V_i satisfying the condition (8.5) in the form $V_i = V_{1i} + V_{2i}$. Put

$$V_{1i} = x_i f(x_i), \qquad V_{2i} = \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} f(x_k) \right)^2,$$

where the numbers α_{ij} are defined in the following way:

$$\alpha_{ij} = \frac{a_{ij} - a_{i,j-1}}{2 - ac_2}, \quad i \in Z, \ j = 0, 1, \dots, i+1. \tag{8.56}$$

Here it is supposed that $a_{i,-1} = 0$, $a_{i,i+1} = a$.

From (8.52)–(8.54) and (8.56) it follows that the numbers α_{ij} satisfy the conditions

$$0 \le \alpha_{i+1,j} \le \alpha_{ij}, \quad i \in Z, \ j = 0, 1, \dots, i+1. \tag{8.57}$$

Using (8.51) we have

$$\Delta V_{1i} = -x_i f(x_i) + x_{i+1} f(x_{i+1}) \le -c_1 x_i^2 + x_{i+1} f(x_{i+1}).$$

Representing ΔV_{2i} in the form

$$\begin{aligned}
\Delta V_{2i} &= \sum_{j=0}^{i+1} \alpha_{i+1,j} \left(\sum_{k=j}^{i+1} f(x_k) \right)^2 - \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} f(x_k) \right)^2 \\
&= \sum_{j=0}^{i+1} (\alpha_{i+1,j} - \alpha_{ij}) \left(\sum_{k=j}^{i+1} f(x_k) \right)^2 \\
&\quad + \sum_{j=0}^{i} \alpha_{ij} \left[\left(\sum_{k=j}^{i+1} f(x_k) \right)^2 - \left(\sum_{k=j}^{i} f(x_k) \right)^2 \right] + \alpha_{i,i+1} f^2(x_{i+1})
\end{aligned}$$

and using (8.57) we get

$$\begin{aligned}
\Delta V_{2i} &\le \sum_{j=0}^{i} \alpha_{ij} \left[f^2(x_{i+1}) + 2 f(x_{i+1}) \sum_{k=j}^{i} f(x_k) \right] + \alpha_{i,i+1} f^2(x_{i+1}) \\
&= f^2(x_{i+1}) \sum_{j=0}^{i+1} \alpha_{ij} + 2 f(x_{i+1}) \sum_{k=0}^{i} f(x_k) \sum_{j=0}^{k} \alpha_{ij}.
\end{aligned}$$

From (8.56) it follows that

$$\sum_{j=0}^{k} \alpha_{ij} = \sum_{j=0}^{k} \frac{a_{ij} - a_{i,j-1}}{2 - ac_2} = \frac{a_{ik}}{2 - ac_2}.$$

Therefore, using (8.51) and (8.50), by $x_{i+1} \ne 0$ we obtain

$$\begin{aligned}
\Delta V_{2i} &\le \frac{af^2(x_{i+1})}{2 - ac_2} + \frac{2 f(x_{i+1})}{2 - ac_2} \sum_{k=0}^{i} a_{ik} f(x_k) \\
&= \frac{a}{2 - ac_2} \frac{f(x_{i+1})}{x_{i+1}} x_{i+1} f(x_{i+1}) + \frac{2 f(x_{i+1})}{2 - ac_2} (\eta_{i+1} - x_{i+1}) \\
&\le \frac{ac_2}{2 - ac_2} x_{i+1} f(x_{i+1}) + \frac{2 f(x_{i+1})(\eta_{i+1} - x_{i+1})}{2 - ac_2} \\
&= -x_{i+1} f(x_{i+1}) + \frac{2 f(x_{i+1}) \eta_{i+1}}{2 - ac_2}.
\end{aligned}$$

As a result for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\mathbf{E}\Delta V_i \le -c_1 \mathbf{E} x_i^2 + \frac{2}{2 - ac_2} \mathbf{E} f(x_{i+1}) \eta_{i+1}. \tag{8.58}$$

Using (8.51), (8.50) and (8.55) and some $\lambda > 0$ we get

$$\begin{aligned}
\left|\mathbf{E} f(x_{i+1})\eta_{i+1}\right| &\le c_2 \mathbf{E}|x_{i+1}\eta_{i+1}| \\
&\le c_2 \mathbf{E}\left(\eta_{i+1}^2 + \sum_{j=0}^{i} a_{ij}\left|f(x_j)\eta_{i+1}\right|\right) \\
&\le c_2\left(\mathbf{E}\eta_{i+1}^2 + c_2 \sum_{j=0}^{i} a_{ij}\mathbf{E}|x_j\eta_{i+1}|\right) \\
&\le c_2\left(\mathbf{E}\eta_{i+1}^2 + \frac{c_2}{2}\sum_{j=0}^{i} a_{ij}\left(\lambda \mathbf{E}x_j^2 + \frac{1}{\lambda}\mathbf{E}\eta_{i+1}^2\right)\right) \\
&\le c_2\left(1 + \frac{\alpha_1 c_2}{2\lambda}\right)\mathbf{E}\eta_{i+1}^2 + \frac{c_2^2\lambda}{2}\sum_{j=0}^{i} a_{ij}\mathbf{E}x_j^2. \qquad (8.59)
\end{aligned}$$

Substituting (8.59) into (8.58), we obtain

$$\mathbf{E}\Delta V_i \le -c_1 \mathbf{E}x_i^2 + \sum_{j=0}^{i} Q_{ij}\mathbf{E}x_j^2 + \frac{c_2(2\lambda + \alpha_1 c_2)}{\lambda(2 - ac_2)}\mathbf{E}\eta_{i+1}^2,$$

where

$$Q_{ij} = \frac{c_2^2 \lambda a_{ij}}{2 - ac_2}, \qquad \sum_{i=j}^{\infty} Q_{ij} \le \frac{c_2^2\lambda\alpha_2}{2 - ac_2}.$$

Since $\alpha_2 < \infty$ then there exists small enough $\lambda > 0$ such that $c_1 > c_2^2\lambda\alpha_2(2 - ac_2)^{-1}$. Thus, via Corollary 1.1 the solution of (8.50) is mean square summable. The proof is completed. □

Remark 8.11 In the case $a_{ij} = a_{i-j}$, the conditions (8.52)–(8.55) have the form

$$\begin{aligned}
&a_i \ge a_{i+1} \ge 0, \qquad a_{i+2} - 2a_{i+1} + a_i \ge 0, \quad i \in Z, \\
&2a_0 - a_1 < \frac{2}{c_2}, \qquad \alpha_1 = \alpha_2 = \sum_{j=0}^{\infty} a_j < \infty.
\end{aligned} \qquad (8.60)$$

Remark 8.12 If the kernel a_{ij} in (8.50) is a degenerate one, i.e. $a_{ij} = p_i q_j$, then the conditions (8.52)–(8.55) take the form

$$0 \le p_{i+1} \le p_i, \qquad 0 \le q_j \le q_{j+1},$$

$$a = \sup_{i \in Z} \big[p_{i+1}(q_{i+1} - q_i) + p_i q_i \big] < \frac{2}{c_2},$$

$$\alpha_1 = \sup_{i \in Z} \left(p_i \sum_{j=0}^{i} q_j \right) \infty, \qquad \alpha_2 = \sup_{j \in Z} \left(q_j \sum_{i=j}^{\infty} p_i \right) < \infty.$$

Remark 8.13 Note that without loss of generality in the condition (8.51) we can put $c_1 \le c_2 = 1$. In fact, if it is not so we can put for instance $a_{ij} f(x_j) = \tilde{a}_{ij} \tilde{f}(x_j)$, where $\tilde{a}_{ij} = c_2 a_{ij}$, $\tilde{f}(x) = c_2^{-1} f(x)$. In this case the function $\tilde{f}(x)$ satisfies the condition (8.51) with $c_2 = 1$.

Example 8.5 Let in (8.50) $a_{ij} = a_{i-j}$ and $a_i = \lambda q^i$, $i \in Z$, $\lambda > 0$, $0 < q < 1$. From Theorem 8.6 using Remarks 8.9 and 8.13 for comparing the estimations on $f(x)$ from (8.46) and (8.51) we obtain a sufficient condition for mean square summability of the solution of (8.50)

$$\lambda c_2 + q < 1. \tag{8.61}$$

From Theorem 8.7 and (8.60) there follows another sufficient condition for the mean square summability of the solution of (8.50):

$$\lambda c_2 < \frac{2}{2 - q}. \tag{8.62}$$

It is easy to see that the condition (8.62) is weaker than (8.61), since $1 - q < 2(2 - q)^{-1}$.

Example 8.6 Consider (8.50) with $a_{ij} = a_{i-j}$ and $a_i = \lambda(i + 1)^{-\gamma}$, $\lambda > 0$, $\gamma > 1$, $i \in Z$. In this case the conditions (8.60) hold and $\alpha_1 = \alpha_2 = \lambda\zeta(\gamma)$, where $\zeta(\gamma)$ is Riemann function,

$$\zeta(\gamma) = \sum_{i=1}^{\infty} \frac{1}{i^\gamma} < \infty.$$

From Theorem 8.6 and Remark 8.9 we obtain a sufficient condition for mean square summability of the solution of (8.50)

$$\lambda c_2 < \zeta^{-1}(\gamma). \tag{8.63}$$

From Theorem 8.7 and Remark 8.13 we obtain another condition:

$$\lambda c_2 < \frac{2}{2 - 2^{-\gamma}}. \tag{8.64}$$

It is easy to see that $\zeta^{-1}(\gamma) < 1$, but $2(2 - 2^{-\gamma})^{-1} > 1$. Thus, condition (8.64) is weaker than (8.63). For instance, by $\gamma = 2$ the conditions (8.63) and (8.64) take the form

$$\lambda c_2 < \zeta^{-1}(2) = 1.645^{-1} = 0.608, \qquad \lambda c_2 < \frac{2}{2 - 2^{-2}} = \frac{8}{7} = 1.143.$$

Example 8.7 Consider (8.50) with

$$a_{ij} = \frac{\lambda(j+1)^{\gamma}}{(i+2)^{\gamma+1}}, \quad 0 \le j \le i, \ \lambda > 0, \ \gamma > 0. \tag{8.65}$$

Similar to Example 8.2 we obtain

$$\alpha_1 \le \frac{\lambda}{\gamma+1}, \qquad \alpha_2 \le \frac{\lambda}{\gamma}.$$

From (8.47) by $p = 2$ using Remark 8.13 we obtain a sufficient condition for the mean square summability in the form

$$\lambda c_2 < \sqrt{\gamma(\gamma+1)}. \tag{8.66}$$

Using (8.54) and (8.65) we have

$$\begin{aligned} a &= \lambda \sup_{i \in Z} \left(\frac{(i+2)^{\gamma} - (i+1)^{\gamma}}{(i+3)^{\gamma+1}} + \frac{(i+1)^{\gamma}}{(i+2)^{\gamma+1}} \right) \\ &\le 2\lambda \sup_{x \ge 0} \left(\frac{x^{\gamma}}{(x+1)^{\gamma+1}} \right) = \frac{2\lambda\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}. \end{aligned} \tag{8.67}$$

From this via (8.54) and Remark 5.7 another summability condition follows:

$$\lambda c_2 < \frac{(\gamma+1)^{\gamma+1}}{\gamma^{\gamma}}. \tag{8.68}$$

In spite of the fact that the estimate (8.67) is rough enough, the condition (8.68) is weaker than (8.66). Actually for given $\gamma > 0$ it is possible to get an estimate that is essentially weaker than (8.67) and as a result to get a summability condition that is weaker than (8.68). For instance, by $\gamma = 1$ it is easy to see that

$$a = \lambda \sup_{i \in Z} \left(\frac{1}{(i+3)^2} + \frac{i+1}{(i+2)^2} \right) \le \frac{13\lambda}{36} \tag{8.69}$$

and the summability conditions (8.66), (8.68) and (8.54) with the estimation (8.69) take the forms, respectively, of

$$\lambda c_2 < \sqrt{2} = 1.414, \qquad \lambda c_2 < 4, \qquad \lambda c_2 < \frac{72}{13} = 5.538.$$

If $\gamma = 2$ then

$$a = \lambda \sup_{i \in Z} \left(\frac{2i+3}{(i+3)^3} + \frac{(i+1)^2}{(i+2)^3} \right) \le \frac{17\lambda}{72} \tag{8.70}$$

and the summability conditions (8.66), (8.68) and (8.54) with estimation (8.70) take the forms, respectively,

$$\lambda c_2 < \sqrt{6} = 2.449, \qquad \lambda c_2 < 6.75, \qquad \lambda c_2 < \frac{144}{17} = 8.471.$$

So, a sufficient condition for the mean square summability of the solution of (8.50) given by Theorem 8.7 is better than the similar condition given by Theorem 8.6.

8.5.4 Resolvent Representation

Consider the nonlinear difference equation

$$x_{i+1} = \eta_{i+1} + \sum_{j=0}^{i} f_{ij}(x_j), \tag{8.71}$$

Theorem 8.8 *Let the function $f_{ij}(x)$ satisfy the inequality*

$$\left| f_{ij}(x) \right| \leq a_{ij} |x| \tag{8.72}$$

and let the kernel a_{ij} have the resolvent b_{ij} such that

$$\beta_1 = \sup_{i \in Z} \sum_{j=0}^{i} b_{ij} < \infty.$$

Then the solution of (8.71) is mean square stable. If besides

$$\beta_2 = \sup_{j \in Z} \sum_{i=j}^{\infty} b_{ij} < \infty$$

then the solution of (8.71) is mean square summable for each mean square summable η_i, $i \in Z$.

Proof From (8.71) and (8.72) it follows that

$$|x_{i+1}| \leq |\eta_{i+1}| + \sum_{j=0}^{i} a_{ij} |x_j|. \tag{8.73}$$

Let y_i be the solution of the equation

$$y_{i+1} = |\eta_{i+1}| + \sum_{j=0}^{i} a_{ij} y_j. \tag{8.74}$$

Then from Lemma 8.1 it follows that y_{i+1} can be represented by

$$y_{i+1} = |\eta_{i+1}| + \sum_{j=0}^{i} b_{ij} |\eta_j|.$$

Let us prove that $|x_i| \le y_i$ with probability 1. Put $y_i^{(0)} = |x_i|$,

$$y_{i+1}^{(n+1)} = |\eta_{i+1}| + \sum_{j=0}^{i} a_{ij} y_j^{(n)}, \quad n = 0, 1, \ldots. \tag{8.75}$$

Show that $y_i^{(0)} \le y_i^{(1)}$. In fact, from (8.73) and (8.75) we get

$$y_{i+1}^{(0)} = |x_{i+1}| \le |\eta_{i+1}| + \sum_{j=0}^{i} a_{ij}|x_j| = |\eta_{i+1}| + \sum_{j=0}^{i} a_{ij} y_j^{(0)} = y_{i+1}^{(1)}.$$

Similar one can show that $|x_i| = y_i^{(0)} \le y_i^{(1)} \le \cdots \le y_i^{(n)} \le \cdots$. Therefore, there exists $\lim_{n\to\infty} y_i^{(n)} = y_i$, and y_i is the solution of (8.74). Hence $|x_i| \le y_i$ and from mean square stability and mean square summability of the solution of (8.74) (Theorem 8.2) it follows that the solution of (8.71) is mean square stable and mean square summable too. The proof is completed. □

From Theorems 8.8 and 8.3 it follows that the next theorem is true too.

Theorem 8.9 *Let the function $f_{ij}(x_j)$ in (8.71) satisfy inequality (8.72) and the inequality*

$$\alpha_1 = \sup_{i\ge 0} \sum_{j=0}^{i} a_{ij} < 1$$

holds. Then the solution of (8.71) is mean square stable. If besides

$$\alpha_2 = \sup_{j\in Z} \sum_{i=j}^{\infty} a_{ij} < 1$$

then the solution of (8.71) is mean square summable for each mean square summable $\eta_i, i \in Z$.

Similarly, Corollary 8.2 can be proven

Corollary 8.3 *Suppose that the function $f_{ij}(x)$ in (8.71) satisfies the condition $|f_{ij}(x)| \le p_i' q_j |x|$, where $p_i' = (p_1^{(i)}, \ldots, p_n^{(i)})$, $q_j' = (q_1^{(j)}, \ldots, q_n^{(j)})$. Then the conditions (8.34) and (8.35) are sufficient conditions for the mean square stability and mean square summability of the solution of (8.71).*

Example 8.8 Consider the nonlinear difference equation

$$x_{i+1} = \eta_{i+1} + \lambda \sum_{j=0}^{i} \sin\left[\frac{(j+1)^\gamma}{(i+2)^{\gamma+1}} x_j\right], \quad \gamma > 0. \tag{8.76}$$

Since

$$\left|\sin\left[\frac{(j+1)^{\gamma}}{(i+2)^{\gamma+1}}\right]x\right| \le \frac{(j+1)^{\gamma}}{(i+2)^{\gamma+1}}|x|$$

then via Theorem 8.9 the conditions for the mean square stability and mean square summability of the solution of (8.76) are obtained in Example 8.2.

Chapter 9
Difference Equations with Continuous Time

In this chapter some of the results, obtained in previous chapters, are repeated for stochastic difference equations with continuous time that are popular enough with researchers [27, 59, 146, 177, 179, 200, 201, 205, 238–241, 243]. It is shown that after some modification of the basic Lyapunov type theorem the general method of the construction of the Lyapunov functionals can be used also for difference equations with continuous time. Also some peculiarities of the investigation of difference equations with continuous time and their differences from difference equations with discrete time are shown.

9.1 Preliminaries and General Statements

9.1.1 Notations, Definitions and Lyapunov Type Theorem

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space and $\{\mathfrak{F}_t, t \geq t_0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e. $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$ for $t_1 < t_2$, $\mathbf{E}$ be the mathematical expectation with respect to the measure $\mathbf{P}$, and let $\mathbf{E}_t = \mathbf{E}\{./\mathfrak{F}_t\}$ be the conditional expectation.

Consider a stochastic difference equation

$$\begin{aligned} x(t+\tau) &= a_1\big(t, x(t), x(t-h_1), x(t-h_2), \dots\big) \\ &\quad + a_2\big(t, x(t), x(t-h_1), x(t-h_2), \dots\big)\xi(t+\tau), \qquad t > t_0 - \tau, \end{aligned} \tag{9.1}$$

with the initial condition

$$x(\theta) = \phi(\theta), \qquad \theta \in \Theta = [t_0 - h, t_0],\ h = \tau + \max_{j \geq 1} h_j. \tag{9.2}$$

Here $x \in \mathbf{R}^n$, $\tau, h_1, h_2, \dots$ are positive constants, the functionals $a_1 \in \mathbf{R}^n$ and $a_2 \in \mathbf{R}^{n \times m}$ satisfy the condition

$$\big|a_l(t, x_0, x_1, x_2, \dots)\big|^2 \leq \sum_{j=0}^{\infty} a_{lj}|x_j|^2, \qquad A = \sum_{l=1}^{2}\sum_{j=0}^{\infty} a_{lj} < \infty, \tag{9.3}$$

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6_9, © Springer-Verlag London Limited 2011

$\phi(\theta)$, $\theta \in \Theta$, is a $\mathfrak{F}_{t_0}$-measurable function, and the perturbation $\xi(t) \in \mathbf{R}^m$ is a $\mathfrak{F}_t$-measurable stationary stochastic process such that $\xi(t)$ is independent on $\mathfrak{F}_s$ for $s \leq t - \tau$,

$$\mathbf{E}_t \xi(t+\tau) = 0, \qquad \mathbf{E}_t \xi(t+\tau)\xi'(t+\tau) = I, \quad t > t_0 - \tau. \tag{9.4}$$

A solution of problems (9.1) and (9.2) is a $\mathfrak{F}_t$-measurable process $x(t) = x(t; t_0, \phi)$, that is equal to the initial function $\phi(t)$ from (9.2) for $t \leq t_0$ and with probability 1 is defined by (9.1) for $t > t_0$.

Definition 9.1 The solution of (9.1) with initial condition (9.2) is called uniformly mean square bounded if there exists a positive number C such that for all $t \geq t_0$

$$\mathbf{E}\big|x(t; t_0, \phi)\big|^2 \leq C. \tag{9.5}$$

Definition 9.2 The trivial solution of (9.1), (9.2) is called mean square stable if for any $\epsilon > 0$ and t_0 there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\mathbf{E}|x(t; t_0, \phi)|^2 < \epsilon$ for all $t \geq t_0$ if $\|\phi\|^2 = \sup_{\theta \in \Theta} \mathbf{E}|\phi(\theta)|^2 < \delta$.

Definition 9.3 The solution of (9.1) with initial condition (9.2) is called asymptotically mean square trivial if

$$\lim_{t \to \infty} \mathbf{E}\big|x(t; t_0, \phi)\big|^2 = 0. \tag{9.6}$$

Definition 9.4 The solution of (9.1) with initial condition (9.2) is called asymptotically mean square quasitrivial if for each $t \in [t_0, t_0 + \tau)$

$$\lim_{j \to \infty} \mathbf{E}\big|x(t + j\tau; t_0, \phi)\big|^2 = 0. \tag{9.7}$$

Definition 9.5 The trivial solution of (9.1), (9.2) is called asymptotically mean square stable if it is mean square stable and for each initial function ϕ the solution of (9.1) is asymptotically mean square trivial.

Definition 9.6 The trivial solution of (9.1), (9.2) is called asymptotically mean square quasistable if it is mean square stable and for each initial function ϕ the solution of (9.1) is asymptotically mean square quasitrivial.

Definition 9.7 The solution of (9.1) with initial condition (9.2) is called uniformly mean square summable if

$$\sup_{t \in [t_0, t_0 + \tau)} \sum_{j=0}^{\infty} \mathbf{E}\big|x(t + j\tau; t_0, \phi)\big|^2 < \infty. \tag{9.8}$$

Definition 9.8 The solution of (9.1) with initial condition (9.2) is called mean square integrable if

$$\int_{t_0}^{\infty} \mathbf{E}\big|x(t;t_0,\phi)\big|^2 \, dt < \infty. \tag{9.9}$$

Remark 9.1 If the solution of (9.1) and (9.2) is asymptotically mean square trivial then it is also asymptotically mean square quasitrivial but the inverse statement is not true. In fact, let us construct the function $y(t)$, $t \geq t_0$, by the following way. On the interval $[t_0 + n\tau, t_0 + (n+1)\tau)$, $n = 0, 1, \ldots$, put

$$y(t) = 0$$

if

$$t \in \left[t_0 + n\tau, t_0 + \left(n + 1 - \frac{1}{2^n}\right)\tau\right)$$

or

$$t \in \left[t_0 + \left(n + 1 - \frac{1}{2^{n+1}}\right)\tau, t_0 + (n+1)\tau\right),$$

put

$$y(t) = 2^{n+2}\left[\frac{t - t_0}{\tau} - n - 1 + \frac{1}{2^n}\right]$$

if

$$t \in \left[t_0 + \left(n + 1 - \frac{1}{2^n}\right)\tau, t_0 + \left(n + 1 - \frac{3}{2^{n+2}}\right)\tau\right)$$

and put

$$y(t) = 1 - 2^{n+2}\left[\frac{t - t_0}{\tau} - n - 1 + \frac{3}{2^{n+2}}\right]$$

if

$$t \in \left[t_0 + \left(n + 1 - \frac{3}{2^{n+2}}\right)\tau, t_0 + \left(n + 1 - \frac{1}{2^{n+1}}\right)\tau\right).$$

The graph of the function $y(t)$ for $t_0 = 0$, $\tau = 1$ is shown in Fig. 9.1. This function satisfies the conditions:

$$0 \leq y(t) \leq 1, \quad \sum_{j=0}^{\infty} y(t + j\tau) \leq 1, \quad \int_0^{\infty} y(t) \, dt = \frac{1}{2}.$$

It is easy to see also that for each fixed $t \in [t_0, t_0 + \tau)$ the sequence $a_j = y(t + j\tau)$ has only one nonzero member and therefore $\lim_{j\to\infty} y(t + j\tau) = 0$. On the other

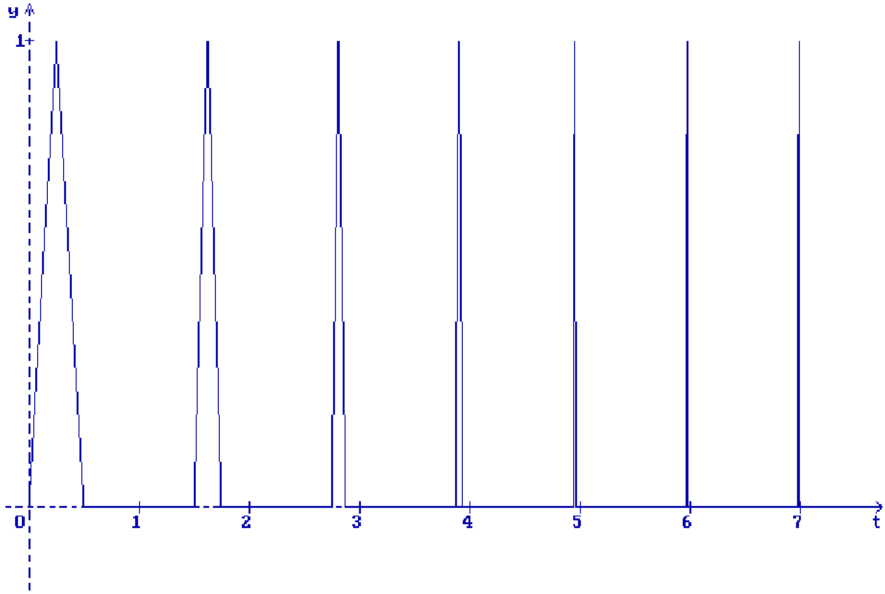


Fig. 9.1 The graph of the function $y(t)$ for $t_0 = 0$, $\tau = 1$ (see Remark 9.1)

hand for every $T > 0$ there exists enough large number n such that

$$t_1 = t_0 + \left(n + 1 - \frac{3}{2^{n+2}}\right)\tau > T, \qquad y(t_1) = 1.$$

Therefore, $\lim_{t\to\infty} y(t)$ does not exist. So, the function $y(t)$ satisfies the condition (9.7) but does not satisfy condition (9.6).

Remark 9.2 From the condition (9.8) it follows that

$$\sup_{t \ge t_0} \sum_{j=0}^{\infty} \mathbf{E}\big|x(t + j\tau; t_0, \phi)\big|^2 < \infty.$$

In fact, arbitrary $t \ge t_0$ can be represented in the form $t = s + k\tau$ with an integer $k \ge 0$ and $s \in [t_0, t_0 + \tau)$. So,

$$\begin{aligned}\sum_{j=0}^{\infty} \mathbf{E}\big|x(t + j\tau; t_0, \phi)\big|^2 &= \sum_{j=0}^{\infty} \mathbf{E}\big|x\big(s + (k + j)\tau; t_0, \phi\big)\big|^2 \\ &= \sum_{j=k}^{\infty} \mathbf{E}\big|x(s + j\tau; t_0, \phi)\big|^2 \\ &\le \sup_{s \in [t_0, t_0+\tau)} \sum_{j=0}^{\infty} \mathbf{E}\big|x(s + j\tau; t_0, \phi)\big|^2 < \infty.\end{aligned}$$

Remark 9.3 If the solution of (9.1) and (9.2) is uniformly mean square summable then it is uniformly mean square bounded and asymptotically mean square quasitrivial.

Below it is supposed that the functional $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$ equals zero if and only if $x(t) = x(t-h_1) = x(t-h_2) = \dots = 0$. Also let $[t]$ be the integer part of a number t and

$$\Delta V(t) = V(t+\tau) - V(t). \tag{9.10}$$

Theorem 9.1 *Let there exist a nonnegative functional $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$ and positive numbers c_1, c_2, such that*

$$\mathbf{E}V(s) \le c_1 \sup_{\theta \le s} \mathbf{E}\big|\phi(\theta)\big|^2, \quad s \in (t_0 - \tau, t_0], \tag{9.11}$$

$$\mathbf{E}\Delta V(t) \le -c_2 \mathbf{E}\big|x(t)\big|^2, \quad t \ge t_0. \tag{9.12}$$

Then the trivial solution of (9.1) and (9.2) is asymptotically mean square quasistable.

Proof Rewrite the condition (9.12) in the form

$$\mathbf{E}\Delta V(t + j\tau) \le -c_2 \mathbf{E}\big|x(t + j\tau)\big|^2, \quad t \ge t_0, \; j = 0, 1, \dots.$$

Summing this inequality from $j = 0$ to $j = i$, by virtue of (9.10) we obtain

$$\mathbf{E}V\big(t + (i+1)\tau\big) - \mathbf{E}V(t) \le -c_2 \sum_{j=0}^{i} \mathbf{E}\big|x(t + j\tau)\big|^2.$$

Therefore,

$$c_2 \sum_{j=0}^{\infty} \mathbf{E}\big|x(t + j\tau)\big|^2 \le \mathbf{E}V(t), \quad t \ge t_0. \tag{9.13}$$

Via (9.10) and (9.12) we have

$$\begin{aligned} &\mathbf{E}V(t) \le \mathbf{E}V(t-\tau) \le \mathbf{E}V(t-2\tau) \le \dots \le \mathbf{E}V(s), \\ &t \ge t_0, \; s = t - q(t)\tau \in (t_0 - \tau, t_0], \end{aligned} \tag{9.14}$$

where

$$q(t) = \begin{cases} [\frac{t-t_0}{\tau}] & \text{if } [\frac{t-t_0}{\tau}] = \frac{t-t_0}{\tau}, \\ [\frac{t-t_0}{\tau}] + 1 & \text{if } [\frac{t-t_0}{\tau}] < \frac{t-t_0}{\tau}. \end{cases} \tag{9.15}$$

From (9.13), (9.14) and (9.11) it follows that

$$c_2 \sum_{j=0}^{\infty} \mathbf{E}\big|x(t + j\tau)\big|^2 \le c_1 \|\phi\|^2, \quad t \ge t_0, \tag{9.16}$$

and also

$$c_2\mathbf{E}\big|x(t)\big|^2 \le c_1\|\phi\|^2, \quad t \ge t_0. \tag{9.17}$$

From (9.17) we see that the trivial solution of (9.1), (9.2) is mean square stable. From (9.16) it follows that the solution of (9.1) and (9.2) is uniformly mean square summable and therefore asymptotically mean square quasitrivial. So, the trivial solution of (9.1) and (9.2) is asymptotically mean square quasistable. Theorem is proven. □

Remark 9.4 If the conditions of Theorem 9.1 hold and $A < 1$ (A is defined in (9.3)) then the trivial solution of (9.1) and (9.2) is asymptotically mean square stable. In fact, via (9.1)–(9.4) for $t \ge t_0$ we obtain

$$\begin{aligned}
\mathbf{E}\big|x(t)\big|^2 &= \mathbf{E}\big|a_1\big(t-\tau, x(t-\tau), x(t-\tau-h_1), x(t-\tau-h_2), \dots\big) \\
&\quad + a_2\big(t-\tau, x(t-\tau), x(t-\tau-h_1), x(t-\tau-h_2), \dots\big)\xi(t)\big|^2 \\
&= \sum_{l=1}^{2} \mathbf{E}\big|a_l\big(t-\tau, x(t-\tau), x(t-\tau-h_1), x(t-\tau-h_2), \dots\big)\big|^2 \\
&\le \sum_{l=1}^{2} \Big(a_{l0}\mathbf{E}\big|x(t-\tau)\big|^2 + \sum_{j=1}^{\infty} a_{lj}\mathbf{E}\big|x(t-\tau-h_j)\big|^2\Big) \\
&\le A \sup_{s \le t-\tau} \mathbf{E}\big|x(s)\big|^2.
\end{aligned} \tag{9.18}$$

Repeating inequality (9.18) we obtain

$$\mathbf{E}\big|x(t)\big|^2 \le A^{q(t)}\|\phi\|^2, \quad t \ge t_0, \tag{9.19}$$

where $q(t)$ is defined by (9.15). Via $A < 1$ and $\lim_{t\to\infty} q(t) = \infty$ we have

$$\lim_{t\to\infty} \mathbf{E}\big|x(t)\big|^2 = 0$$

for each initial function ϕ. So, the trivial solution of (9.1) and (9.2) is asymptotically mean square stable.

Remark 9.5 If the conditions of Theorem 9.1 hold then the solution of (9.1) for each initial function (9.2) is mean square summable and mean square integrable. In fact, summability follows from (9.16). Besides integrating (9.12) from $t = t_0$ to $t = T$, by virtue of (9.10) we have

$$\int_{t_0}^{T} \mathbf{E}\big(V(t+\tau) - V(t)\big)\,\mathrm{d}t \le -c_2 \int_{t_0}^{T} \mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t$$

or

$$\int_{T}^{T+\tau} \mathbf{E}V(t)\,\mathrm{d}t - \int_{t_0}^{t_0+\tau} \mathbf{E}V(t)\,\mathrm{d}t \le -c_2 \int_{t_0}^{T} \mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t.$$

From this and (9.14) and (9.11) it follows that

$$c_2 \int_{t_0}^{T} \mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t \le \int_{t_0}^{t_0+\tau} \mathbf{E}V(t)\,\mathrm{d}t \le \int_{t_0-\tau}^{t_0} \mathbf{E}V(s)\,\mathrm{d}s \le c_1\tau\|\phi\|^2 < \infty$$

and by $T \to \infty$ we obtain (9.9).

Corollary 9.1 *Let there exist a functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$ *and positive numbers* c_1, c_2, *such that the conditions* (9.11), $\mathbf{E}V(t) \ge c_2\mathbf{E}|x(t)|^2$ *and* $\mathbf{E}\Delta V(t) \le 0$ *hold. Then the trivial solution of* (9.1) *is mean square stable.*

Corollary 9.2 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$, *which satisfies the conditions* (9.11) *and* $\mathbf{E}\Delta V(t) = -c\mathbf{E}|x(t)|^2$, $t \ge t_0$. *Then the inequality* $c > 0$ *is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of* (9.1).

Proof A sufficiency follows from Theorem 9.1. To prove a necessity it is enough to note that if $c \le 0$ then

$$\sum_{j=0}^{i} \mathbf{E}\Delta V(t+j\tau) = \mathbf{E}V(t+i\tau) - \mathbf{E}V(t) \ge 0$$

or $\mathbf{E}V(t+i\tau) \ge \mathbf{E}V(t) > 0$. It means that the trivial solution of (9.1) cannot be asymptotically mean square quasistable.

From Theorem 9.1 and Corollaries 9.1, 9.2 it follows that an investigation of stability of the trivial solution of (9.1) can be reduced to the construction of appropriate Lyapunov functionals. Below some formal procedure of the construction of the Lyapunov functionals for an equation of the type of (9.1) is described. □

9.1.2 Formal Procedure of the Construction of the Lyapunov Functionals

The proposed procedure of the construction of the Lyapunov functionals consists of four steps.

Step 1. Represent the functionals a_1 and a_2 at the right-hand side of (9.1) in the form

$$\begin{aligned} a_1\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) &= F_1(t) + F_2(t) + \Delta F_3(t), \\ a_2\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) &= G_1(t) + G_2(t), \end{aligned} \tag{9.20}$$

where

$$F_1(t) = F_1\big(t, x(t), x(t-h_1), \dots, x(t-h_k)\big),$$

$$F_j(t) = F_j\big(t, x(t), x(t-h_1), x(t-h_2), \dots\big), \quad j = 2, 3,$$

$$F_1(t, 0, \dots, 0) \equiv F_2(t, 0, 0, \dots) \equiv F_3(t, 0, 0, \dots) \equiv 0,$$

$$G_1(t) = G_1\big(t, x(t), x(t-h_1), \dots, x(t-h_k)\big),$$

$$G_2(t) = G_2\big(t, x(t), x(t-h_1), x(t-h_2), \dots\big),$$

$$G_1(t, 0, \dots, 0) \equiv G_2(t, 0, 0, \dots) \equiv 0,$$

$k \geq 0$ is a given integer, $\Delta F_3(t) = F_3(t+\tau) - F_3(t)$.

Step 2. Suppose that for the auxiliary equation

$$y(t+\tau) = F_1\big(t, x(t), x(t-h_1), \dots, x(t-h_k)\big) + G_1\big(t, x(t), x(t-h_1), \dots, x(t-h_k)\big)\xi(t+\tau), \quad t > t_0 - \tau, \tag{9.21}$$

there exists a Lyapunov functional $v(t) = v(t, y(t), y(t-h_1), \dots, y(t-h_k))$, that satisfies the conditions of Theorem 9.1.

Step 3. Consider Lyapunov functional $V(t)$ for (9.1) in the form $V(t) = V_1(t) + V_2(t)$, where the main component is

$$V_1(t) = v\big(t, x(t) - F_3(t), x(t-h_1), \dots, x(t-h_k)\big).$$

It is necessary to calculate $\mathbf{E}\Delta V_1(t)$ and in a reasonable way to estimate it.

Step 4. In order to satisfy the conditions of Theorem 9.1 the additional component $V_2(t)$ is chosen by some standard way.

Note that some standard way for construction of additional functional V_2 allows to simplify the fourth step of the procedure and do not use the functional V_2 at all. Below corresponding auxiliary Lyapunov type theorems are considered.

9.1.3 Auxiliary Lyapunov Type Theorems

The following theorems in some cases allow one to construct Lyapunov functionals with conditions that are weaker than (9.12).

Theorem 9.2 *Let there exist a nonnegative functional* $V_1(t) = V_1(t, x(t), x(t-h_1), x(t-h_2), \dots)$, *which satisfies the condition* (9.11) *and the conditions*

$$\mathbf{E}\Delta V_1(t) \le a\mathbf{E}\big|x(t)\big|^2 + \sum_{j=1}^{N(t)} A(t, t - j\tau)\mathbf{E}\big|x(t - j\tau)\big|^2,$$
$$N(t) = \left[\frac{t+h}{\tau}\right], \quad A(t,s) \ge 0, \quad s \le t,\ t \ge t_0, \tag{9.22}$$

$$a + b < 0, \quad b = \sup_{t \ge t_0} \sum_{j=1}^{\infty} A(t + j\tau, t). \tag{9.23}$$

Then the trivial solution of (9.1) *and* (9.2) *is asymptotically mean square quasistable.*

Proof According to the procedure of the construction of the Lyapunov functionals described above, let us consider the functional $V(t)$ in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t)$ satisfies the conditions (9.22) and (9.23) and

$$V_2(t) = \sum_{m=1}^{N(t)} \big|x(t - m\tau)\big|^2 \sum_{j=m}^{\infty} A\big(t + (j-m)\tau, t - m\tau\big).$$

Note that $N(t+\tau) = N(t) + 1$. So, calculating $\mathbf{E}\Delta V_2(t)$, we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_2(t) = {} & \sum_{m=1}^{N(t)+1} \mathbf{E}\big|x(t + \tau - m\tau)\big|^2 \\
& \times \sum_{j=m}^{\infty} A\big(t + \tau + (j-m)\tau, t + \tau - m\tau\big) - \mathbf{E}V_2(t) \\
= {} & \mathbf{E}\big|x(t)\big|^2 \sum_{j=1}^{\infty} A(t + j\tau, t) + \sum_{m=2}^{N(t)+1} \mathbf{E}\big|x(t + \tau - m\tau)\big|^2 \\
& \times \sum_{j=m}^{\infty} A\big(t + \tau + (j-m)\tau, t + \tau - m\tau\big) - \mathbf{E}V_2(t) \\
= {} & \mathbf{E}\big|x(t)\big|^2 \sum_{j=1}^{\infty} A(t + j\tau, t) + \sum_{k=1}^{N(t)} \mathbf{E}\big|x(t - k\tau)\big|^2 \\
& \times \sum_{j=k+1}^{\infty} A\big(t + (j-k)\tau, t - k\tau\big) \\
& - \sum_{m=1}^{N(t)} \mathbf{E}\big|x(t - m\tau)\big|^2 \sum_{j=m}^{\infty} A\big(t + (j-m)\tau, t - m\tau\big) \\
= {} & \mathbf{E}\big|x(t)\big|^2 \sum_{j=1}^{\infty} A(t + j\tau, t) - \sum_{m=1}^{N(t)} A(t, t - m\tau)\mathbf{E}\big|x(t - m\tau)\big|^2.
\end{aligned}$$

From this and (9.22) and (9.23), for the functional $V(t) = V_1(t) + V_2(t)$ we get $\mathbf{E}\Delta V(t) \le (a+b)\mathbf{E}|x(t)|^2$. Together with (9.23) this inequality implies (9.12). So, there exists the functional $V(t)$, that satisfies the conditions of Theorem 9.1, i.e. the trivial solution of (9.1) and (9.2) is asymptotically mean square quasistable. The theorem is proven. □

Theorem 9.3 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$, *that satisfies the conditions* (9.11) *and*

$$\mathbf{E}\Delta V(t) \le a\mathbf{E}\left|x(t)\right|^2 + b\mathbf{E}\left|x(t-k\tau)\right|^2, \quad t \ge t_0, \tag{9.24}$$

where k is a positive integer. If the solution of (9.1) *and* (9.2) *is uniformly mean square bounded but is not uniformly mean square summable then*

$$a + b \ge 0. \tag{9.25}$$

Proof Rewrite (9.24) for $t + j\tau$ with $t \ge t_0$, $j = 0, 1, \ldots$, i.e.

$$\mathbf{E}\Delta V(t+j\tau) \le a\mathbf{E}\left|x(t+j\tau)\right|^2 + b\mathbf{E}\left|x\big(t+(j-k)\tau\big)\right|^2. \tag{9.26}$$

Summing (9.26) from $j = 0$ to $j = i + k$, we obtain

$$\begin{aligned}
&\mathbf{E}V\big(t+(i+k+1)\tau\big) - \mathbf{E}V(t) \\
&\quad \le a\sum_{j=0}^{i+k}\mathbf{E}\left|x(t+j\tau)\right|^2 + b\sum_{j=0}^{i+k}\mathbf{E}\left|x(t+(j-k)\tau)\right|^2 \\
&\quad = a\sum_{j=0}^{i+k}\mathbf{E}\left|x(t+j\tau)\right|^2 + b\sum_{j=0}^{i}\mathbf{E}\left|x(t+j\tau)\right|^2 + b\sum_{j=-k}^{-1}\mathbf{E}\left|x(t+j\tau)\right|^2 \\
&\quad = (a+b)\sum_{j=0}^{i}\mathbf{E}\left|x(t+j\tau)\right|^2 + a\sum_{j=i+1}^{i+k}\mathbf{E}\left|x(t+j\tau)\right|^2 + b\sum_{j=-k}^{-1}\mathbf{E}\left|x(t+j\tau)\right|^2.
\end{aligned}$$

From this and $V(t) \ge 0$ it follows that

$$\begin{aligned}
-(a+b)\sum_{j=0}^{i}\mathbf{E}\left|x(t+j\tau)\right|^2 &\le \mathbf{E}V(t) + a\sum_{j=i+1}^{i+k}\mathbf{E}\left|x(t+j\tau)\right|^2 \\
&\quad + b\sum_{j=-k}^{-1}\mathbf{E}\left|x(t+j\tau)\right|^2, \quad t \ge t_0.
\end{aligned} \tag{9.27}$$

Consider $t \in (t_0 - \tau, t_0]$. Since the solution of (9.1) and (9.2) is uniformly mean square bounded, i.e. $\mathbf{E}|x(t)|^2 \le C$, then using (9.11) and (9.2), we have

$$-(a+b)\sum_{j=0}^{i}\mathbf{E}\big|x(t+j\tau)\big|^2 \le c_1\|\phi\|^2 + k\big(|a|C + |b|\|\phi\|^2\big) < \infty.$$

Let us suppose that (9.25) does not hold, i.e. $a+b<0$. Then the condition (9.8) holds, i.e. the solution of (9.1) and (9.2) is uniformly mean square summable, and we obtain a contradiction with the condition of Theorem 9.3. Therefore, (9.25) holds. The theorem is proven. □

Corollary 9.3 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$ *that satisfies the conditions* (9.11) *and* (9.24) *and*

$$a+b<0. \tag{9.28}$$

Then the solution of (9.1) *and* (9.2) *is either mean square unbounded or uniformly mean square summable.*

Corollary 9.4 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$, *that satisfies conditions* $\mathbf{E}V(t) \le c_1 \sup_{s\le t}\mathbf{E}|x(s)|^2$, $t \ge t_0$, *and* (9.24). *If the solution of* (9.1) *and* (9.2) *is uniformly mean square bounded but is not mean square integrable then the condition* (9.25) *holds. If the condition* (9.28) *holds then the solution of* (9.1) *and* (9.2) *is either mean square unbounded or mean square integrable.*

Proof Integrating (9.24) from $t=t_0$ to $t=T$, we have

$$\begin{aligned}
\int_{t_0}^{T}\mathbf{E}\Delta V(t)\,\mathrm{d}t &= \int_{T}^{T+\tau}\mathbf{E}V(t)\,\mathrm{d}t - \int_{t_0}^{t_0+\tau}\mathbf{E}V(t)\,\mathrm{d}t \\
&\le a\int_{t_0}^{T}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t + b\int_{t_0-k\tau}^{T-k\tau}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t \\
&= (a+b)\int_{t_0}^{T}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t + b\int_{t_0-k\tau}^{t_0}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t \\
&\quad - b\int_{T-k\tau}^{T}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t.
\end{aligned}$$

Using $V(t) \ge 0$ and (9.2), we obtain

$$\begin{aligned}
&-(a+b)\int_{t_0}^{T}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t \\
&\quad\le \int_{t_0}^{t_0+\tau}\mathbf{E}V(t)\,\mathrm{d}t + b\int_{t_0-k\tau}^{t_0}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t - b\int_{T-k\tau}^{T}\mathbf{E}\big|x(t)\big|^2\,\mathrm{d}t \\
&\quad\le \tau\big[c_1C + k|b|\big(\|\phi\|^2 + C\big)\big].
\end{aligned}$$

The statement of Corollary 9.4 follows from this similarly to Theorem 9.3 and Corollary 9.3. □

Theorem 9.4 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$, *that satisfies the conditions* (9.11), (9.24) *and* (9.28). *If* $a > 0$ *then each uniformly mean square bounded solution of* (9.1) *is asymptotically mean square quasitrivial. If* $a \le 0$ *then the trivial solution of* (9.1) *is asymptotically mean square quasistable.*

Proof It follows from Corollary 9.3 that by the conditions (9.11), (9.24) and (9.28) each uniformly mean square bounded solution of (9.1) is uniformly mean square summable and, therefore, it is asymptotically mean square quasitrivial. Let us show that if $a \le 0$ then the trivial solution of (9.1) is stable. In fact, if $a = 0$ then from (9.28) we have $b < 0$. So it follows from (9.24) that

$$\mathbf{E}\Delta V(t) \le b\mathbf{E}\big|x(t-k\tau)\big|^2 \le 0. \tag{9.29}$$

Rewrite condition (9.29) in the form $\mathbf{E}\Delta V(t+j\tau) \le b\mathbf{E}|x(t+(j-k)\tau)|^2$, $t \ge t_0$, $j = 0, 1, \dots$. Summing this inequality from $j = 0$ to $j = k$, we obtain

$$\mathbf{E}V\big(t+(k+1)\tau\big) - \mathbf{E}V(t) \le b\sum_{j=0}^{k}\mathbf{E}\big|x\big(t+(j-k)\tau\big)\big|^2.$$

Therefore,

$$|b|\mathbf{E}\big|x(t)\big|^2 \le |b|\sum_{j=0}^{k}\mathbf{E}\big|x\big(t+(j-k)\tau\big)\big|^2 \le \mathbf{E}V(t), \quad t \ge t_0. \tag{9.30}$$

It follows also from (9.29) that

$$\mathbf{E}V(t) \le \mathbf{E}V(t-\tau) \le \mathbf{E}V(t-2\tau) \le \dots \le \mathbf{E}V(s), \quad t \ge t_0, \tag{9.31}$$

where $s = t - q(t)\tau \in (t_0 - \tau, t_0])$, $q(t)$ is defined by (9.15). From (9.30), (9.31) and (9.11) we obtain $|b|\mathbf{E}|x(t)|^2 \le c_1\|\phi\|^2$, $t \ge t_0$. It means that the trivial solution of (9.1) and (9.2) is mean square stable.

Let $a < 0$. If $b \le 0$ then the condition (9.11) follows from (9.24). So, it follows from Theorem 9.1 that the trivial solution of (9.1) is asymptotically mean square quasistable. If $b > 0$ then the condition (9.24) is a particular case of (9.22). It follows from this and (9.11) and (9.28) that the functional $V(t)$ satisfies the conditions of Theorem 9.2 and, therefore, the trivial solution of (9.1) and (9.2) is asymptotically mean square quasistable. The theorem is proven. □

Corollary 9.5 *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$, *that satisfies the conditions* (9.11) *and*

$$\mathbf{E}\Delta V(t) = a\mathbf{E}\big|x(t)\big|^2 + b\mathbf{E}\big|x(t-k\tau)\big|^2, \quad b > 0, \ t \ge t_0.$$

Then inequality (9.28) *is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of* (9.1).

Proof Sufficiency follows from Theorem 9.4 and necessity from Corollary 9.2. □

Example 9.1 Consider the equation

$$x(t+1) = \alpha x(t) + \beta x(t-k) + \sigma x(t-m)\xi(t+1). \tag{9.32}$$

In compliance with the procedure of the construction of the Lyapunov functionals let us consider an auxiliary equation in the form $y(t+1) = \alpha y(t)$. If $|\alpha| < 1$ then the functional $v(t) = y^2(t)$ is a Lyapunov functional for this equation, since $\Delta v(t) = y^2(t+1) - y^2(t) = (\alpha^2 - 1)y^2(t)$.

Put $V_1(t) = x^2(t)$. Calculating $\mathbf{E}\Delta V_1(t)$ for (9.32), we have

$$\begin{aligned}
\mathbf{E}\Delta V_1(t) &= \mathbf{E}\big[x^2(t+1) - x^2(t)\big] \\
&= \mathbf{E}\big[\alpha x(t) + \beta x(t-k) + \sigma x(t-m)\xi(t+1)\big]^2 - \mathbf{E}x^2(t) \\
&= \big(\alpha^2 - 1\big)\mathbf{E}x^2(t) + 2\alpha\beta\mathbf{E}x(t)x(t-k) + \beta^2\mathbf{E}x^2(t-k) + \sigma^2\mathbf{E}x^2(t-m) \\
&\le \big(\alpha^2 + |\alpha\beta| - 1\big)\mathbf{E}x^2(t) + \big(|\alpha\beta| + \beta^2\big)\mathbf{E}x^2(t-k) + \sigma^2\mathbf{E}x^2(t-m).
\end{aligned} \tag{9.33}$$

Choosing an additional functional $V_2(t)$ in the form

$$V_2(t) = \big(|\alpha\beta| + \beta^2\big)\sum_{j=1}^{k} x^2(t-j) + \sigma^2\sum_{j=1}^{m} x^2(t-j),$$

we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_2(t) &= \big(|\alpha\beta| + \beta^2\big)\sum_{j=1}^{k}\mathbf{E}\big[x^2(t+1-j) - x^2(t-j)\big] \\
&\quad + \sigma^2\sum_{j=1}^{m}\mathbf{E}\big[x^2(t+1-j) - x^2(t-j)\big] \\
&= \big(|\alpha\beta| + \beta^2\big)\mathbf{E}\big[x^2(t) - x^2(t-k)\big] + \sigma^2\mathbf{E}\big[x^2(t) - x^2(t-m)\big] \\
&= \big(|\alpha\beta| + \beta^2 + \sigma^2\big)\mathbf{E}x^2(t) - \big(|\alpha\beta| + \beta^2\big)\mathbf{E}x^2(t-k) - \sigma^2\mathbf{E}x^2(t-m).
\end{aligned} \tag{9.34}$$

It follows from (9.33) and (9.34) that if the inequality $(|\alpha| + |\beta|)^2 + \sigma^2 < 1$ holds then the functional $V(t) = V_1(t) + V_2(t)$ satisfies the conditions of Theorem 9.1 and, therefore, the trivial solution of (9.32) is asymptotically mean square quasistable.

By virtue of Theorem 9.2 the same result can be obtained via the functional $V_1(t)$ only without construction of the additional functional $V_2(t)$. In fact, it follows from (9.33) that the functional $V_1(t)$ satisfies the conditions (9.22) and (9.23) with $a=\alpha^2+|\alpha\beta|-1$ and $b=|\alpha\beta|+\beta^2+\sigma^2$.

From Corollary 9.5 it follows that if $\beta=0$ then the inequality $\alpha^2+\sigma^2<1$ is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.32). It is easy to show also that if $\alpha=0$ then the inequality $\beta^2+\sigma^2<1$ is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.32).

Remark 9.6 Suppose that in (9.1) $h_j=j\tau$, $j=1,2,\dots$. Putting $t=t_0+s\tau$, $y(s)=x(t_0+s\tau)$, $\eta(s)=\xi(t_0+s\tau)$, one can reduce (9.1) to the form

$$\begin{aligned} y(s+1)=&b_1\big(s,y(s),y(s-1),y(s-2),\dots\big)\\ &+b_2\big(s,y(s),y(s-1),y(s-2),\dots\big)\eta(s+1),\quad s>-1,\\ y(\theta)=&\phi(\theta),\quad \theta\le 0. \end{aligned}\tag{9.35}$$

Below, the equations type of (9.35) are considered.

9.2 Linear Volterra Equations with Constant Coefficients

Let us demonstrate the formal procedure of the construction of the Lyapunov functionals described above for stability investigation of the scalar equation

$$\begin{aligned} x(t+1)&=\sum_{j=0}^{[t]+r}a_jx(t-j)+\sum_{j=0}^{[t]+r}\sigma_jx(t-j)\xi(t+1),\quad t>-1,\\ x(s)&=\phi(s),\quad s\in\big[-(r+1),0\big], \end{aligned}\tag{9.36}$$

where $r\ge 0$ is a given integer, a_j and σ_j are known constants.

9.2.1 First Way of the Construction of the Lyapunov Functional

Following Step 1 of the procedure represent (9.36) in form (9.36) with $F_3(t)=0$, $G_1(t)=0$,

$$F_1(t) = \sum_{j=0}^{k} a_j x(t-j), \qquad F_2(t) = \sum_{j=k+1}^{[t]+r} a_j x(t-j), \quad k \geq 0,$$
$$G_2(t) = \sum_{j=0}^{[t]+r} \sigma_j x(t-j), \tag{9.37}$$

and consider (Step 2) the auxiliary equation

$$y(t+1) = \sum_{j=0}^{k} a_j y(t-j), \quad t > -1, \ k \geq 0,$$
$$y(s) = \begin{cases} \phi(s), & s \in [-(r+1), 0], \\ 0, & s < -(r+1). \end{cases} \tag{9.38}$$

Introduce into consideration the vector $Y(t) = (y(t-k), \dots, y(t-1), y(t))'$ and represent the auxiliary equation (9.38) in the form

$$Y(t+1) = AY(t), \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \dots & a_1 & a_0 \end{pmatrix}. \tag{9.39}$$

Consider the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \tag{9.40}$$

and suppose that the solution D of this equation is a positive semidefinite symmetric matrix of dimension $k+1$ with $d_{k+1,k+1} > 0$. In this case the function $v(t) = Y'(t)DY(t)$ is Lyapunov function for (9.39), i.e. it satisfies the conditions of Theorem 9.1; in particular, the condition (9.12). In fact, using (9.39) and (9.40), we have

$$\begin{aligned} \Delta v(t) &= Y'(t+1)DY(t+1) - Y'(t)Dy(t) \\ &= Y'(t)\big[A'DA - D\big]Y(t) = -Y'(t)UY(t) = -y^2(t). \end{aligned}$$

Following Step 3 of the procedure, we will construct the Lyapunov functional $V(t)$ for (9.36) in the form $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = X'(t)DX(t), \quad X(t) = \big(x(t-k), \dots, x(t-1), x(t)\big)'. \tag{9.41}$$

Rewrite now (9.36) using the representation (9.37)

$$\begin{aligned} X(t+1) &= AX(t)+B(t), \\ B(t) &= \big(0,\dots,0,b(t)\big)', \ b(t)=F_2(t)+G_2(t)\xi(t+1), \end{aligned} \tag{9.42}$$

where the matrix A is defined by (9.39). Calculating $\Delta V_1(t)$, by virtue of (9.42) we have

$$\begin{aligned} \Delta V_1(t) &= X'(t+1)DX(t+1)-X'(t)DX(t) \\ &= \big(AX(t)+B(t)\big)'D\big(AX(t)+B(t)\big)-X'(t)DX(t) \\ &= -x^2(t)+B'(t)DB(t)+2B'(t)DAX(t). \end{aligned} \tag{9.43}$$

Put

$$\alpha_l=\sum_{j=l}^{\infty}|a_j|, \qquad \delta_l=\sum_{j=l}^{\infty}|\sigma_j|, \quad l=0,1,\dots. \tag{9.44}$$

Using (9.42), (9.37) and (9.44), we obtain

$$\begin{aligned} &\mathbf{E}B'(t)DB(t) \\ &\quad= d_{k+1,k+1}\mathbf{E}b^2(t) \\ &\quad= d_{k+1,k+1}\mathbf{E}\big[F_2(t)+G_2(t)\xi(t+1)\big]^2 = d_{k+1,k+1}\big[\mathbf{E}F_2^2(t)+\mathbf{E}G_2^2(t)\big] \\ &\quad= d_{k+1,k+1}\left[\mathbf{E}\left(\sum_{j=k+1}^{[t]+r}a_jx(t-j)\right)^2+\mathbf{E}\left(\sum_{j=0}^{[t]+r}\sigma_jx(t-j)\right)^2\right] \\ &\quad\le d_{k+1,k+1}\left[\alpha_{k+1}\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)+\delta_0\sum_{j=0}^{[t]+r}|\sigma_j|\mathbf{E}x^2(t-j)\right]. \end{aligned} \tag{9.45}$$

Since

$$DB(t)=b(t)\begin{pmatrix} d_{1,k+1} \\ d_{2,k+1} \\ \dots \\ d_{k,k+1} \\ d_{k+1,k+1}\end{pmatrix}, \qquad AX(t)=\begin{pmatrix} x(t-k+1) \\ x(t-k+2) \\ \dots \\ x(t) \\ \sum_{m=0}^k a_mx(t-m)\end{pmatrix},$$

then

$$\begin{aligned} &\mathbf{E}B'(t)DAX(t) \\ &\quad= \mathbf{E}b(t)\left[\sum_{l=1}^{k}d_{l,k+1}x(t-k+l)+d_{k+1,k+1}\sum_{m=0}^{k}a_mx(t-m)\right] \end{aligned}$$

$$= \mathbf{E}b(t)\left[\sum_{m=0}^{k-1}(a_m d_{k+1,k+1} + d_{k-m,k+1})x(t-m) + a_k d_{k+1,k+1}x(t-k)\right]$$

$$= d_{k+1,k+1}\mathbf{E}F_2(t)\sum_{m=0}^{k} Q_{km}x(t-m), \tag{9.46}$$

where

$$Q_{km} = a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}, \quad m = 0,\dots,k-1, \qquad Q_{kk} = a_k. \tag{9.47}$$

Putting

$$\beta_k = \sum_{m=0}^{k}|Q_{km}| = |a_k| + \sum_{m=0}^{k-1}\left|a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}\right| \tag{9.48}$$

and using (9.46), (9.37), (9.44) and (9.48), we obtain

$$\begin{aligned}
&2\mathbf{E}B'(t)DAX(t)\\
&= 2d_{k+1,k+1}\sum_{m=0}^{k}\sum_{j=k+1}^{[t]+r} Q_{km}a_j\mathbf{E}x(t-m)x(t-j)\\
&\le d_{k+1,k+1}\sum_{m=0}^{k}\sum_{j=k+1}^{[t]+r}|Q_{km}||a_j|\big(\mathbf{E}x^2(t-m) + \mathbf{E}x^2(t-j)\big)\\
&\le d_{k+1,k+1}\sum_{m=0}^{k}|Q_{km}|\left(\alpha_{k+1}\mathbf{E}x^2(t-m) + \sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)\right)\\
&= d_{k+1,k+1}\left(\alpha_{k+1}\sum_{m=0}^{k}|Q_{km}|\mathbf{E}x^2(t-m) + \beta_k\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)\right).
\end{aligned} \tag{9.49}$$

Now put

$$R_{km} = \begin{cases}\alpha_{k+1}|Q_{km}| + \delta_0|\sigma_m|, & 0 \le m \le k,\\ (\alpha_{k+1} + \beta_k)|a_m| + \delta_0|\sigma_m|, & m > k.\end{cases} \tag{9.50}$$

Then from (9.43), (9.45) and (9.49) it follows that

$$\mathbf{E}\Delta V_1(t) \le -\mathbf{E}x^2(t) + d_{k+1,k+1}\sum_{m=0}^{[t]+r} R_{km}\mathbf{E}x^2(t-m). \tag{9.51}$$

It means that the functional $V_1(t)$ satisfies condition (9.22). It is easy to see that condition (9.11) holds too. So, via Theorem 9.2 if

$$\gamma_0 d_{k+1,k+1} < 1, \quad \gamma_0 = \sum_{j=0}^{\infty} R_{kj}, \tag{9.52}$$

then the trivial solution of (9.36) is asymptotically mean square quasistable.

Using (9.52), (9.50) and (9.48), transform γ_0 in the following way:

$$\begin{aligned}
\gamma_0 &= \sum_{j=0}^{k} R_{kj} + \sum_{j=k+1}^{\infty} R_{kj} \\
&= \sum_{j=0}^{k} \big(\alpha_{k+1}|Q_{kj}| + \delta_0|\sigma_j|\big) + \sum_{j=k+1}^{\infty} \big((\alpha_{k+1}+\beta_k)|a_j| + \delta_0|\sigma_j|\big) \\
&= \alpha_{k+1}\beta_k + (\alpha_{k+1}+\beta_k)\alpha_{k+1} + \delta_0^2 = \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k + \delta_0^2.
\end{aligned}$$

Thus, if

$$\alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k + \delta_0^2 < d_{k+1,k+1}^{-1}, \tag{9.53}$$

then the trivial solution of (9.36) is asymptotically mean square quasistable.

Note that the condition (9.53) can also be represent in the form

$$\alpha_{k+1} < \sqrt{\beta_k^2 + d_{k+1,k+1}^{-1} - \delta_0^2} - \beta_k. \tag{9.54}$$

Remark 9.7 Suppose that in (9.36)

$$a_j = 0, \quad j > k. \tag{9.55}$$

Then $\alpha_{k+1} = 0$ and the condition (9.53) takes the form

$$\delta_0^2 < d_{k+1,k+1}^{-1}. \tag{9.56}$$

So, if the condition (9.55) holds and matrix equation (9.40) has a positive semidefinite solution D with the condition (9.56), then the trivial solution of (9.36) is asymptotically mean square quasistable.

Remark 9.8 Suppose that in (9.36) condition (9.55) holds and $\sigma_j = 0$, $j \neq m$, for some $m \geq 0$. In this case $\alpha_{k+1} = 0$, $\delta_0^2 = \sigma_m^2$ and from (9.43), (9.45) and (9.49) it follows that

$$\mathbf{E}\Delta V_1(t) = -\mathbf{E}x^2(t) + d_{k+1,k+1}\sigma_m^2 \mathbf{E}x^2(t-m).$$

Via Corollary 9.5 the inequality $d_{k+1,k+1}\sigma_m^2 < 1$ is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.36).

Remark 9.9 From (9.36) and (9.44) it follows that (9.36) satisfies condition (9.3) with $A=\alpha_0^2+\delta_0^2$. So, via Remark 9.4 if

$$\alpha_0^2+\delta_0^2<1 \tag{9.57}$$

then the trivial solution of (9.36) is asymptotically mean square stable.

9.2.2 Second Way of the Construction of the Lyapunov Functional

Let us find another stability condition. Equation (9.36) can be represented (Step 1) in form (9.20) with $F_2(t)=G_1(t)=0$, $k=0$,

$$\begin{aligned} &F_1(t)=\beta x(t), \quad \beta=\sum_{j=0}^{\infty}a_j, \\ &F_3(t)=-\sum_{m=1}^{[t]+r}x(t-m)\sum_{j=m}^{\infty}a_j, \qquad G_2(t)=\sum_{j=0}^{[t]+r}\sigma_j x(t-j). \end{aligned} \tag{9.58}$$

In fact, calculating $\Delta F_3(t)$, we have

$$\begin{aligned} \Delta F_3(t) &= -\sum_{m=1}^{[t]+1+r}x(t+1-m)\sum_{j=m}^{\infty}a_j+\sum_{m=1}^{[t]+r}x(t-m)\sum_{j=m}^{\infty}a_j \\ &= -\sum_{m=0}^{[t]+r}x(t-m)\sum_{j=m+1}^{\infty}a_j+\sum_{m=1}^{[t]+r}x(t-m)\sum_{j=m}^{\infty}a_j \\ &= -x(t)\sum_{j=1}^{\infty}a_j+\sum_{m=1}^{[t]+r}x(t-m)a_m. \end{aligned}$$

Thus,

$$x(t+1)=\beta x(t)+\Delta F_3(t)+G_2(t)\xi(t). \tag{9.59}$$

In this case the auxiliary equation (Step 2) is $y(t+1)=\beta y(t)$. The function $v(t)=y^2(t)$ is Lyapunov function for this equation if $|\beta|<1$. Consider (Step 3) the Lyapunov functional $V_1(t)=(x(t)-F_3(t))^2$. Calculating $\mathbf{E}\Delta V_1(t)$, by virtue of the representation (9.59) we have

$$\begin{aligned} \mathbf{E}\Delta V_1(t) &= \mathbf{E}\big[\big(x(t+1)-F_3(t+1)\big)^2-\big(x(t)-F_3(t)\big)^2\big] \\ &= \mathbf{E}\big[\big(\beta x(t)-F_3(t)+G_2(t)\xi(t)\big)^2-\big(x(t)-F_3(t)\big)^2\big] \\ &= \big(\beta^2-1\big)\mathbf{E}x^2(t)+Q(t), \end{aligned}$$

where

$$Q(t) = -2(\beta - 1)\mathbf{E}x(t)F_3(t) + \mathbf{E}G_2^2(t).$$

Putting

$$\alpha = \sum_{m=1}^{\infty}\left|\sum_{j=m}^{\infty} a_j\right|, \qquad B_m = |\beta - 1|\left|\sum_{j=m}^{\infty} a_j\right| + \delta_0\sigma_m, \tag{9.60}$$

and using (9.58) and (9.44), we obtain

$$\begin{aligned}
\left|Q(t)\right| &\le 2|\beta - 1|\sum_{m=1}^{[t]+r}\left|\sum_{j=m}^{\infty} a_j\right|\mathbf{E}\left|x(t)x(t-m)\right| + \mathbf{E}\left(\sum_{j=0}^{[t]+r}\sigma_j x(t-j)\right)^2 \\
&\le |\beta - 1|\sum_{m=1}^{[t]+r}\left|\sum_{j=m}^{\infty} a_j\right|\left(\mathbf{E}x^2(t) + \mathbf{E}x^2(t-m)\right) \\
&\quad + \sum_{l=0}^{[t]+r}|\sigma_l|\sum_{j=0}^{[t]+r}|\sigma_j|\mathbf{E}x^2(t-j) \\
&\le \left(\alpha|\beta - 1| + \delta_0\sigma_0\right)\mathbf{E}x^2(t) + \sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m).
\end{aligned}$$

As a result we have

$$\mathbf{E}\Delta V_1(t) \le \left(\beta^2 - 1 + \alpha|\beta - 1| + \delta_0\sigma_0\right)\mathbf{E}x^2(t) + \sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m).$$

Via Theorem 9.2 and (9.60) and (9.44) if

$$\beta^2 + 2\alpha|\beta - 1| + \delta_0^2 < 1 \tag{9.61}$$

then the trivial solution of (9.36) is asymptotically mean square quasistable.

It is easy to see that the condition (9.61) can be written also in the form

$$\delta_0^2 < (1-\beta)(1+\beta-2\alpha), \qquad |\beta| < 1. \tag{9.62}$$

9.2.3 Particular Cases and Examples

Consider the particular cases of condition (9.53) (or (9.54)) for different $k \ge 0$.

Case $k = 0$ Equation (9.40) gives the solution $d_{11} = (1 - a_0^2)^{-1}$, which is positive if $|a_0| < 1$. From (9.48) it follows that $\beta_0 = |a_0|$ and the condition (9.53) takes the form (9.57). So by $k = 0$ the condition (9.53) is a sufficient condition for asymptotic mean square stability of the trivial solution of (9.36).

Case $k=1$ The matrix equation (9.40) is equivalent to the system of equations

$$a_1^2 d_{22} - d_{11} = 0,$$
$$(a_1 - 1)d_{12} + a_0 a_1 d_{22} = 0,$$
$$d_{11} + 2a_0 d_{12} + \left(a_0^2 - 1\right)d_{22} = -1$$

with the solution

$$\begin{aligned} &d_{11} = a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1-a_1} d_{22}, \\ &d_{22} = \frac{1-a_1}{(1+a_1)[(1-a_1)^2 - a_0^2]}. \end{aligned} \tag{9.63}$$

The matrix D is positive semidefinite with $d_{22} > 0$ by the conditions $|a_1| < 1$, $|a_0| < 1 - a_1$. Using (9.48) and (9.63), we have

$$\begin{aligned} &\beta_1 = |a_1| + \left| a_0 + \frac{d_{12}}{d_{22}} \right| = |a_1| + \left| a_0 + \frac{a_0 a_1}{1-a_1} \right| = |a_1| + \frac{|a_0|}{1-a_1}, \\ &d_{22}^{-1} = 1 - a_1^2 - a_0^2 \frac{1+a_1}{1-a_1}. \end{aligned} \tag{9.64}$$

Condition (9.54) takes the form

$$\alpha_2 < \sqrt{\left(1 + \frac{|a_0 a_1|}{1-a_1}\right)^2 - \delta_0^2} - |a_1| - \frac{|a_0|}{1-a_1}. \tag{9.65}$$

Under condition (9.65) the trivial solution of (9.36) is asymptotically mean square quasistable.

Note that the condition (9.65) can be written in the form

$$\alpha_0^2 + \delta_0^2 < 1 + \frac{2|a_0|}{1-a_1}\big(|a_1| - \alpha_0 a_1\big). \tag{9.66}$$

It is easy to see that the condition (9.66) is not worse than (9.57).

Case $k=2$ The matrix equation (9.40) is equivalent to the system of equations

$$\begin{aligned} &a_2^2 d_{33} - d_{11} = 0, \\ &a_2 d_{13} + a_1 a_2 d_{33} - d_{12} = 0, \\ &a_2 d_{23} + a_0 a_2 d_{33} - d_{13} = 0, \\ &d_{11} + 2a_1 d_{13} + a_1^2 d_{33} - d_{22} = 0, \\ &d_{12} + a_0 d_{13} + a_0 a_1 d_{33} + (a_1 - 1)d_{23} = 0, \\ &d_{22} + 2a_0 d_{23} + \left(a_0^2 - 1\right)d_{33} = -1 \end{aligned} \tag{9.67}$$

with the solution

$$\begin{aligned}
d_{11} &= a_2^2 d_{33}, \\
d_{12} &= \frac{a_2(1-a_1)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \\
d_{13} &= \frac{a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \\
d_{22} &= \left[a_1^2+a_2^2+\frac{2a_1a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)}\right] d_{33}, \\
d_{23} &= \frac{(a_0+a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \\
d_{33} &= \left[1-a_0^2-a_1^2-a_2^2-2\frac{a_1a_2(a_0+a_1a_2)+a_0(a_0+a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)}\right]^{-1}.
\end{aligned} \tag{9.68}$$

Using (9.48), (9.67) and (9.68), we have

$$\begin{aligned}
\beta_2 &= |a_2| + \left|a_0+\frac{d_{23}}{d_{33}}\right| + \left|a_1+\frac{d_{13}}{d_{33}}\right| = |a_2| + \frac{|d_{13}|+|d_{12}|}{|a_2|d_{33}} \\
&= |a_2| + \frac{|a_0+a_1a_2|+|(1-a_1)(a_1+a_0a_2)|}{|1-a_1-a_2(a_0+a_2)|}.
\end{aligned} \tag{9.69}$$

If the matrix D with the elements defined by (9.68) is positive semidefinite with $d_{33}>0$, then under the condition

$$\alpha_3 < \sqrt{\beta_2^2 + d_{33}^{-1} - \delta_0^2} - \beta_2. \tag{9.70}$$

the solution of (9.36) is asymptotically mean square quasistable.

Example 9.2 Consider the difference equation

$$\begin{aligned}
&x(t+1) = ax(t) + bx(t-1) + \sigma x(t-1)\xi(t+1), \quad t > -1, \\
&x(\theta) = \phi(\theta), \quad \theta \in [-2, 0].
\end{aligned} \tag{9.71}$$

From the conditions (9.57) and (9.65) follow two sufficient conditions for asymptotic mean square quasistability of the trivial solution of (9.71):

$$\big(|a|+|b|\big)^2 + \sigma^2 < 1 \tag{9.72}$$

and

$$\sigma^2 < \big(1-b^2\big)\left(1-\frac{a^2}{(1-b)^2}\right). \tag{9.73}$$

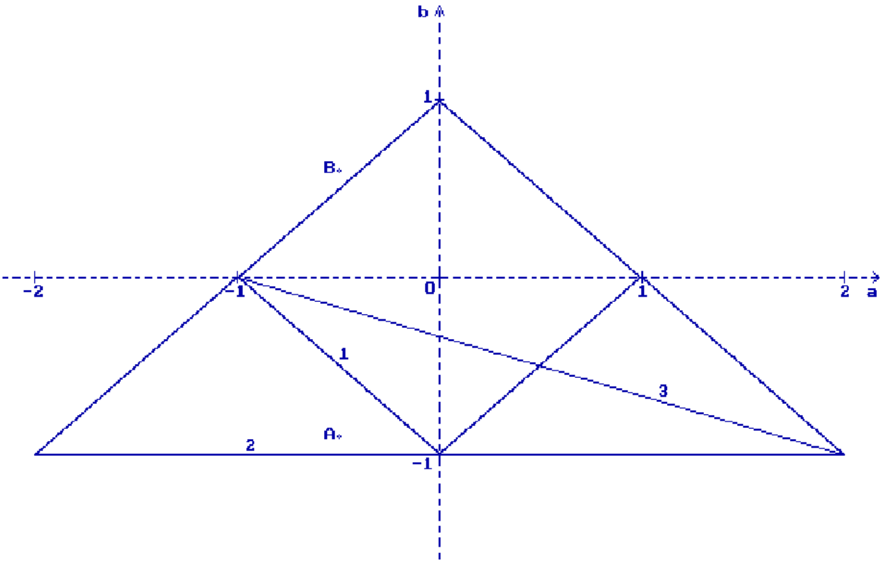


Fig. 9.2 Quasistability regions for (9.71), given for $\sigma^2 = 0$ by the conditions: (*1*) (9.72), (*2*) (9.73), (*3*) (9.74)

Condition (9.70) for (9.71) coincides with (9.73). Condition (9.62) takes the form

$$\sigma^2 < (1 - a - b)\big(1 + a + b - 2|b|\big), \quad |a + b| < 1. \tag{9.74}$$

As follows from Remark 9.9, condition (9.72) is also a sufficient condition for asymptotic mean square stability of the trivial solution of (9.71). From Remark 9.8 it follows that the condition (9.73) is a necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.71).

In Fig. 9.2 the regions of asymptotic mean square quasistability, given by the conditions (9.72) (curve number 1), (9.73) (curve number 2) and (9.74) (curve number 3), are shown for $\sigma^2 = 0$. In Fig. 9.3 a similar picture is shown for $\sigma^2 = 0.2$.

In Fig. 9.4 the solution of (9.71) is shown for $\sigma^2 = 0$ and $\phi(\theta) = \cos 2\theta - 1$ in the point $A(-0.5, -0.9)$, which belongs to the stability region. In Fig. 9.5 the solution of (9.71) is shown for $\sigma^2 = 0$ and $\phi(\theta) = 0.05 \cos 2\theta$ in the point $B(-0.5, 0.6)$, which does not belong to the stability region. The points A and B are shown in Fig. 9.2.

Example 9.3 Consider the difference equation

$$\begin{aligned} &x(t+1) = ax(t) + \sum_{j=1}^{[t]+r} b^j x(t-j) + \sigma x(t-r)\xi(t+1), \quad t > -1, \\ &x(\theta) = \phi(\theta), \quad \theta \in \big[-(r+1), 0\big],\ r \ge 0. \end{aligned} \tag{9.75}$$

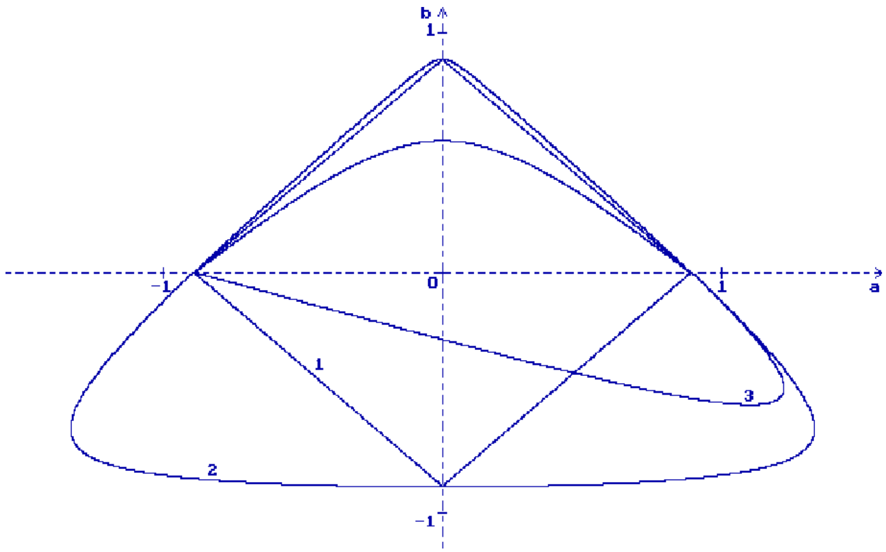


Fig. 9.3 Quasistability regions for (9.71), given for $\sigma^2 = 0.2$ by the conditions: (*1*) (9.72), (*2*) (9.73), (*3*) (9.74)

From the condition (9.57) and Remark 9.9 it follows that the inequality

$$\left(|a| + \frac{|b|}{1-|b|}\right)^2 + \sigma^2 < 1, \quad |b| < 1, \tag{9.76}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (9.75).

Condition (9.65) gives a sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.75) in the form

$$\frac{b^2}{1-|b|} < \sqrt{\left(1 + \frac{|ab|}{1-b}\right)^2 - \sigma^2} - |b| - \frac{|a|}{1-b}, \quad |b| < 1. \tag{9.77}$$

From (9.68)–(9.70) we obtain another sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.75)

$$\begin{aligned} &\frac{|b|^3}{1-|b|} < \sqrt{\beta_2^2 + d_{33}^{-1} - \sigma^2} - \beta_2, \quad |b| < 1, \\ &\beta_2 = b^2 + \frac{|a+b^3| + (1-b)|b(1+ab)|}{|1-b-b^2(a+b^2)|}, \\ &d_{33}^{-1} = 1 - a^2 - b^2 - b^4 - 2b\frac{b^2(a+b^3) + a(a+b^2)(1+ab)}{1-b-b^2(a+b^2)}. \end{aligned} \tag{9.78}$$

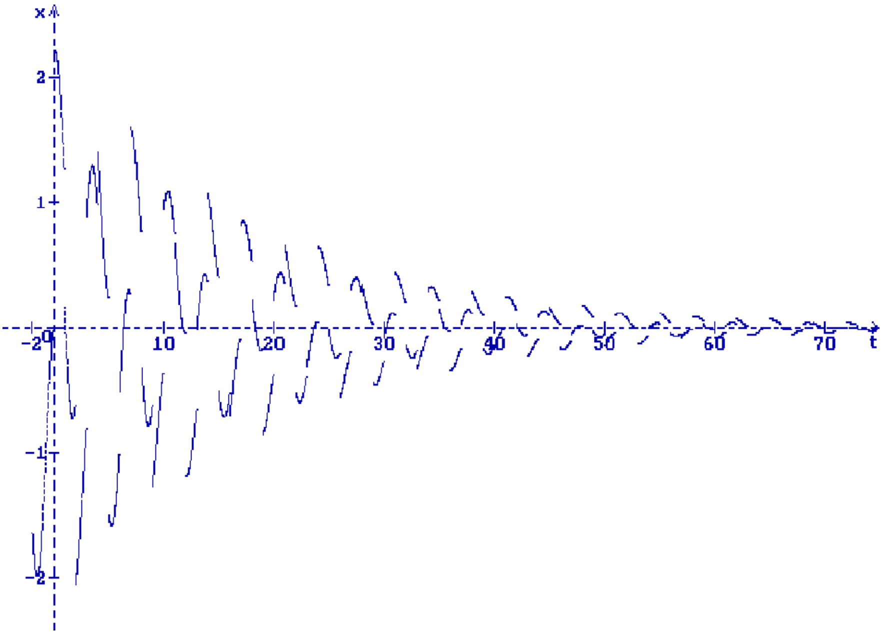


Fig. 9.4 Stable solution of (9.71) in the point $A(-0.5, -0.9)$

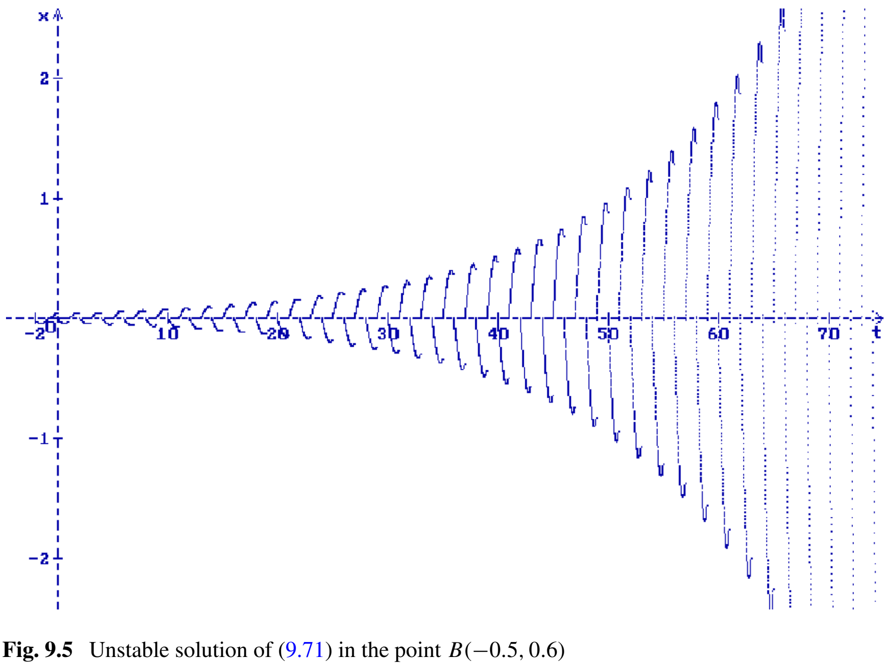


Fig. 9.5 Unstable solution of (9.71) in the point $B(-0.5, 0.6)$

By virtue of the program "Mathematica" the matrix equation (9.40) was solved and a sufficient condition (9.54) for asymptotic mean square quasistability of the trivial solution of (9.75) was obtained also for $k=3$ and $k=4$. In particular, for $k=3$ this condition takes the form

$$\begin{aligned}&\frac{b^4}{1-|b|}<\sqrt{\beta_3^2+d_{44}^{-1}-\sigma^2}-\beta_3, \quad |b|<1,\\&\beta_3=\left|b^3\right|+\left|a+\frac{d_{34}}{d_{44}}\right|+\left|b+\frac{d_{24}}{d_{44}}\right|+\left|b^2+\frac{d_{14}}{d_{44}}\right|,\end{aligned} \tag{9.79}$$

where

$$\frac{d_{14}}{d_{44}}=b^3\big[b^3+b^5-b^8+a\big(1-b^3+b^4\big)\big]G^{-1},$$

$$\frac{d_{24}}{d_{44}}=b^2\big[a^2b+b^2+b^5-b^6-b^8+a\big(1+b^4+b^6\big)\big]G^{-1},$$

$$\frac{d_{34}}{d_{44}}=b\big[b^2+a^3b^2+b^4-b^7+a^2\big(b+b^4\big)+a\big(1+2b^3+b^5-b^6-b^8\big)\big]G^{-1},$$

$$\begin{aligned}d_{44}=G\big[&1-b-b^2-a^4b^3-2b^4+2b^7-2b^8+2b^9-b^{10}-b^{12}+b^{13}\\&-b^{14}+b^{17}-a^3\big(b^2+b^5\big)-a^2\big(1+b+5b^4-b^5+b^6-2b^7-b^9\big)\\&-ab^2\big(1+4b-b^2+5b^3-b^4+b^5-4b^6+4b^7-b^{10}+b^{11}\big)\big]^{-1},\end{aligned}$$

$$G=1-b-ab^2-\big(1+a^2\big)b^3-b^4-ab^5-b^6+b^7+b^9.$$

Condition (9.62) for (9.75) takes the form

$$\begin{aligned}&\sigma^2<(1-\beta)(1+\beta-2\alpha), \quad |\beta|<1,\\&\alpha=\frac{|b|}{(1-b)(1-|b|)}, \qquad \beta=a+\frac{b}{1-b}, \quad |b|<1.\end{aligned} \tag{9.80}$$

In Fig. 9.6 the regions of asymptotic mean square quasistability of the trivial solution of (9.75) for $\sigma^2=0$ are shown, obtained by the condition (9.54) for $k=0$ (condition (9.76), curve number 1), for $k=1$ (condition (9.77), curve number 2), for $k=2$ (condition (9.78), curve number 3), for $k=3$ (condition (9.79), curve number 4), for $k=4$ (curve number 5) and also obtained by the condition (9.80) (curve number 6). In Fig. 9.7 a similar picture is shown for $\sigma^2=0.5$.

As is shown in Fig. 9.6 (and naturally it can be shown analytically) if $\sigma=0$ then for $b\ge 0$ condition (9.76) coincides with the condition (9.77) and for $a\ge 0$, $b\ge 0$ the conditions (9.76)–(9.79) give the same region of asymptotic mean square quasistability, which is defined by the inequality

$$a+\frac{b}{1-b}<1, \quad b<1.$$

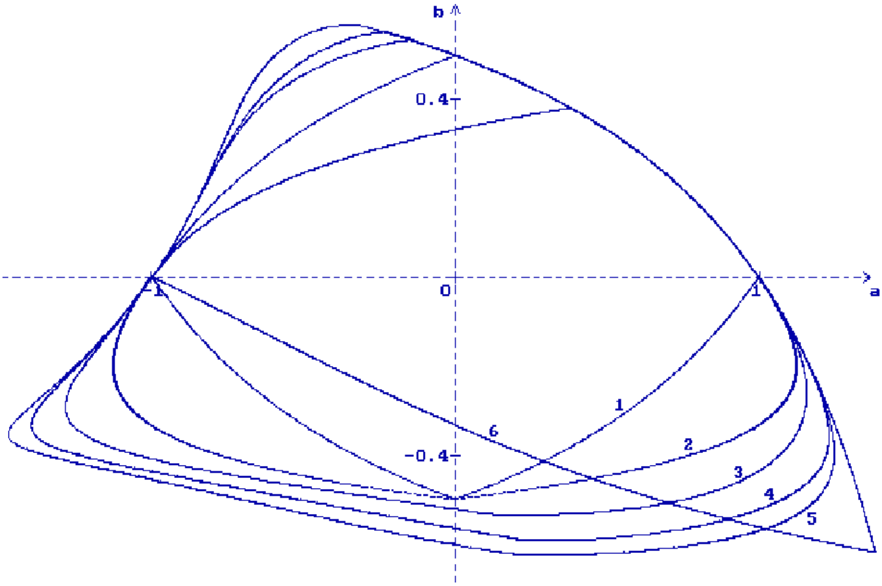


Fig. 9.6 Quasistability regions for (9.75), given for $\sigma^2 = 0$ by different conditions

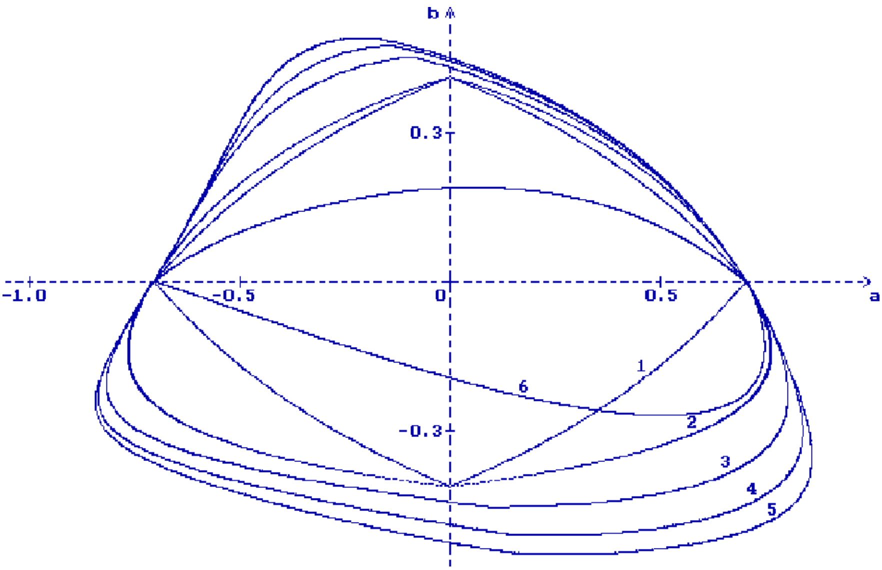


Fig. 9.7 Quasistability regions for (9.75), given for $\sigma^2 = 0.5$ by different conditions

Note also that the region of asymptotic mean square quasistability Q_k of the trivial solution of (9.75), obtained by condition (9.54), expands if k increases, i.e. $Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset Q_4$. So, to get a greater region of asymptotic mean square quasistability one can use the condition (9.54) for $k=5$, $k=6$ etc. But it is clear that each region Q_k can be obtained by the condition $|b|<1$ only.

To obtain a condition for asymptotic mean square quasistability of the trivial solution of (9.75) of another type transform the sum from (9.75) for $t>0$ in the following way:

$$\begin{aligned}\sum_{j=1}^{[t]+r} b^j x(t-j) &= b\sum_{j=1}^{[t]+r} b^{j-1}x(t-j) = b\sum_{j=0}^{[t]-1+r} b^j x(t-1-j)\\ &= b\left(x(t-1)+\sum_{j=1}^{[t]-1+r} b^j x(t-1-j)\right)\\ &= b\big[(1-a)x(t-1)+x(t)-\sigma x(t-1-r)\xi(t)\big]. \end{aligned}\tag{9.81}$$

Substituting (9.81) into (9.75) we obtain (9.75) in the form

$$\begin{aligned} x(t+1) &= ax(t)+\sum_{j=1}^{r-1} b^j x(t-j)+\sigma x(t-r)\xi(t+1), \quad t\in(-1,0],\\ x(t+1) &= (a+b)x(t)+b(1-a)x(t-1)\\ &\quad -b\sigma x(t-1-r)\xi(t)+\sigma x(t-r)\xi(t+1), \quad t>0. \end{aligned}\tag{9.82}$$

Equation (9.82) can be considered in the form (9.1) with $t_0=1$, $\tau=1$,

$$a_1=(a+b)x(t)+b(1-a)x(t-1),$$

$$a_2=\big(-b\sigma x(t-1-r), \sigma x(t-r)\big)\in\mathbf{R}^{1\times 2}$$

and perturbations $(\xi(t),\xi(t+1))'\in\mathbf{R}^2$.

Consider now the functional $V_1(t)$ in the form (9.41) where $k=1$ and the matrix D is defined by (9.63) with

$$a_0=a+b, \qquad a_1=b(1-a). \tag{9.83}$$

Via (9.83) the matrix D is positive semidefinite with $d_{22}>0$ if and only if

$$\big|b(1-a)\big|<1, \qquad |a+b|<1-b(1-a). \tag{9.84}$$

For (9.82) similarly to (9.43) with A and $X(t)$ defined by (9.39) and (9.41) for $k=1$ we obtain

$$\Delta V_1(t)=-x^2(t)+B'(t)DB(t)+2B'(t)DAX(t)$$

with

$$B(t)=\big(0,b(t)\big)', \quad b(t)=\sigma x(t-r)\xi(t+1)-b\sigma x(t-1-r)\xi(t).$$

Similarly to (9.45) we have

$$\mathbf{E}B'(t)DB(t)=d_{22}\mathbf{E}b^2(t)=d_{22}\sigma^2\big(\mathbf{E}x^2(t-r)+b^2\mathbf{E}x^2(t-1-r)\big).$$

Similarly to (9.46) we obtain

$$\begin{aligned}&\mathbf{E}B'(t)DAX(t)\\&\quad=\mathbf{E}b'(t)\big[(a_0d_{22}+d_{12})x(t)+a_1d_{22}x(t-1)\big]\\&\quad=\mathbf{E}\big[\sigma x(t-r)\xi(t+1)-b\sigma x(t-1-r)\xi(t)\big]\\&\qquad\times\big[(a_0d_{22}+d_{12})x(t)+a_1d_{22}x(t-1)\big]\\&\quad=-b\sigma(a_0d_{22}+d_{12})\mathbf{E}x(t-1-r)\xi(t)x(t).\end{aligned}$$

From (9.82) it follows that $\mathbf{E}(\xi(t)x(t)/\mathfrak{F}_t)=\sigma x(t-1-r)$. So,

$$\mathbf{E}B'(t)DAX(t)=-b\sigma^2(a_0d_{22}+d_{12})\mathbf{E}x^2(t-1-r),$$

and by virtue of the representation (9.63)

$$\mathbf{E}B'(t)DAX(t)=-\frac{b\sigma^2a_0d_{22}}{1-a_1}\mathbf{E}x^2(t-1-r).$$

As a result we obtain

$$\mathbf{E}\Delta V_1(t)=-\mathbf{E}x^2(t)+\sigma^2d_{22}\big[\mathbf{E}x^2(t-r)+\gamma\mathbf{E}x^2(t-1-r)\big],\tag{9.85}$$

where, using the representation (9.83),

$$\gamma=b^2-2b\frac{a_0}{1-a_1}=b^2-2b\frac{a+b}{1-b(1-a)}.\tag{9.86}$$

Note that by the condition (9.84) $\gamma>-1$. In fact, using (9.86) and (9.84), we have

$$\gamma+1\geq b^2-2|b|\frac{|a+b|}{1-b(1-a)}+1>b^2-2|b|+1=\big(1-|b|\big)^2\geq 0.$$

Similarly to Corollaries 9.2 and 9.5 from (9.85) it follows that if $\gamma\geq 0$ then the inequality

$$\sigma^2d_{22}(1+\gamma)<1\tag{9.87}$$

is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.82) (or (9.75)).

Besides via Theorem 9.2 from (9.85) it follows that if $\gamma\in(-1,0)$ then the inequality

$$\sigma^2d_{22}<1\tag{9.88}$$

is a sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.82) (or (9.75)).

Let us now consider the case

$$\gamma \in (-1,0), \quad 1 \le \sigma^2 d_{22} < \frac{1}{1+\gamma}. \tag{9.89}$$

Lemma 9.1 *By the condition* (9.89) *each uniformly mean square bounded solution of* (9.82) *is asymptotically mean square quasitrivial.*

Proof In fact, putting in (9.85) $t+j$ instead of t and summing it from $j=0$ to $j=i$ we have

$$\begin{aligned}
&\mathbf{E}V_1(t+i+1) - \mathbf{E}V_1(t)\\
&\quad = -\sum_{j=0}^{i} \mathbf{E}x^2(t+j)\\
&\qquad + \sigma^2 d_{22} \sum_{j=0}^{i} \mathbf{E}x^2(t+j-r) + \sigma^2 d_{22}\gamma \sum_{j=0}^{i} \mathbf{E}x^2(t+j-1-r).
\end{aligned}$$

From this via $V_1(t+i+1) \ge 0$ and $\gamma \in (-1,0)$ it follows that

$$\begin{aligned}
\sum_{j=0}^{i} \mathbf{E}x^2(t+j) &\le \mathbf{E}V_1(t) + \sigma^2 d_{22} \sum_{j=0}^{i} \mathbf{E}x^2(t+j) + \sigma^2 d_{22} \sum_{j=-r}^{-1} \mathbf{E}x^2(t+j)\\
&\quad + \sigma^2 d_{22}\left(\gamma \sum_{j=0}^{i} \mathbf{E}x^2(t+j) + \sum_{j=i-r}^{i} \mathbf{E}x^2(t+j)\right)
\end{aligned}$$

or

$$\begin{aligned}
&\big(1-\sigma^2 d_{22}(1+\gamma)\big) \sum_{j=0}^{i} \mathbf{E}x^2(t+j)\\
&\quad \le \mathbf{E}V_1(t) + \sigma^2 d_{22}\left(\sum_{j=-r}^{-1} \mathbf{E}x^2(t+j) + \sum_{j=i-r}^{i} \mathbf{E}x^2(t+j)\right).
\end{aligned}$$

If the solution of (9.82) is uniformly mean square bounded, i.e. $\mathbf{E}x^2(t) \le C$, then

$$\big(1-\sigma^2 d_{22}(1+\gamma)\big) \sum_{j=0}^{\infty} \mathbf{E}x^2(t+j) \le \mathbf{E}V_1(t) + 2\sigma^2 d_{22}(r+1)C$$

and therefore $\lim_{j\to\infty} \mathbf{E}x^2(t+j) = 0$ for each $t \ge 0$. The proof is completed. □

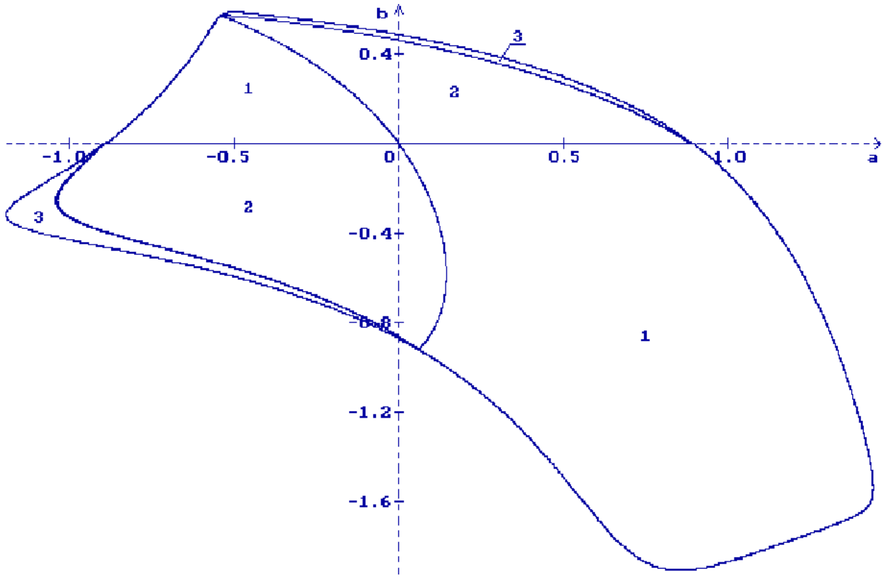


Fig. 9.8 Regions for (9.75) for $\sigma^2 = 0.2$: (*1*) $\{\gamma \geq 0, \sigma^2 d_{22}(1+\gamma) < 1\}$, (*2*) $\{\gamma \in (-1,0), \sigma^2 d_{22} < 1\}$, (*3*) $\{\gamma \in (-1,0), 1 \leq \sigma^2 d_{22} \leq (1+\gamma)^{-1}\}$

So, in the regions $\{\gamma \geq 0, \sigma^2 d_{22}(1+\gamma) < 1\}$ and $\{\gamma \in (-1,0), \sigma^2 d_{22} < 1\}$ the trivial solution of (9.75) is asymptotically mean square quasistable. In the region $\{\gamma \in (-1,0), 1 \leq \sigma^2 d_{22} \leq (1+\gamma)^{-1}\}$ we can conclude only that each uniformly mean square bounded solution of (9.75) is asymptotically mean square trivial. In Fig. 9.8 all these regions are shown for $\sigma^2 = 0.2$: 1) $\{\gamma \geq 0, \sigma^2 d_{22}(1+\gamma) < 1\}$, 2) $\{\gamma \in (-1,0), \sigma^2 d_{22} < 1\}$, 3) $\{\gamma \in (-1,0), 1 \leq \sigma^2 d_{22} \leq (1+\gamma)^{-1}\}$.

In Fig. 9.9 the regions of asymptotic mean square quasistability of the trivial solution of (9.75) for $\sigma^2 = 0.2$ are shown, obtained by the condition (9.54) for $k = 0$ (condition (9.76), curve number 1), for $k = 1$ (condition (9.77), curve number 2), for $k = 2$ (condition (9.78), curve number 3), for $k = 3$ (condition (9.79), curve number 4), for $k = 4$ (curve number 5) and also obtained by condition (9.80) (curve number 6) and condition (9.87) (curve number 7). It is easy to see that some part of the region $\{\gamma \in (-1,0),\ 1 \leq \sigma^2 d_{22} \leq (1+\gamma)^{-1}\}$ belongs to the regions given by the condition (9.54) and therefore the trivial solution of (9.75) is asymptotically mean square quasistable. In Fig. 9.10 by virtue of the condition (9.84) a similar picture is shown for $\sigma^2 = 0$.

In Fig. 9.11 the trajectory of (9.75) is shown for $r = 1$, $\sigma^2 = 0$ and $\phi(\theta) = 0.8\cos\theta$ in the point $A(0.6, -2.3)$, which belongs to the stability region. In Fig. 9.12 the trajectory of (9.75) is shown for $r = 1$, $\sigma^2 = 0$ and $\phi(\theta) = 0.1\cos\theta$ in the point $B(0.54, -2.3)$, which does not belong to the stability region. The points A and B are shown in Fig. 9.9.

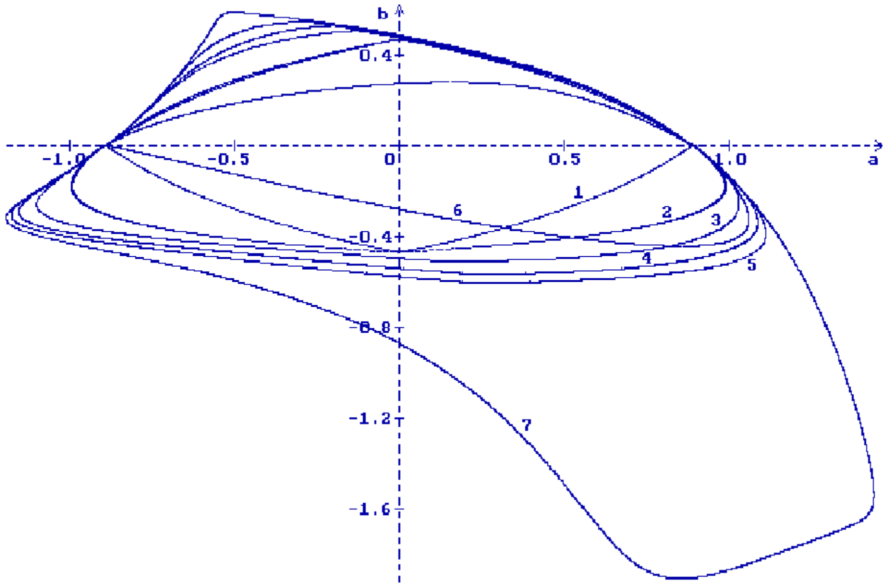


Fig. 9.9 Quasistability regions for (9.75), given for $\sigma^2 = 0.2$ by different conditions

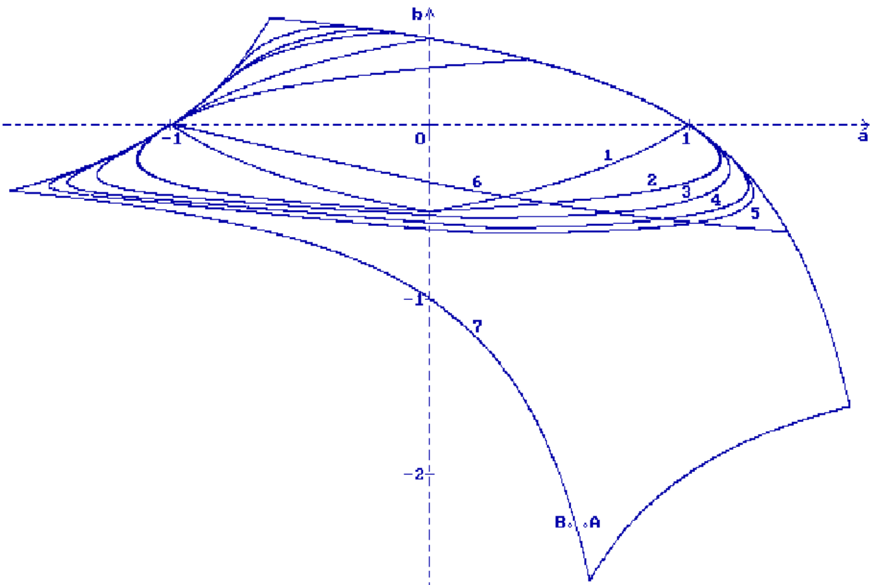


Fig. 9.10 Quasistability regions for (9.75), given for $\sigma^2 = 0$ by different conditions

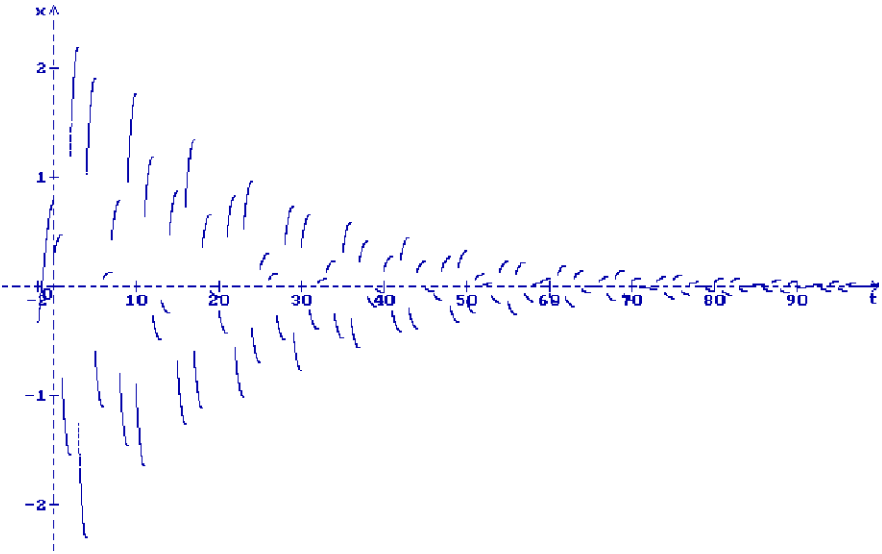


Fig. 9.11 Stable solution of (9.75) in the point $A(0.6, -2.3)$

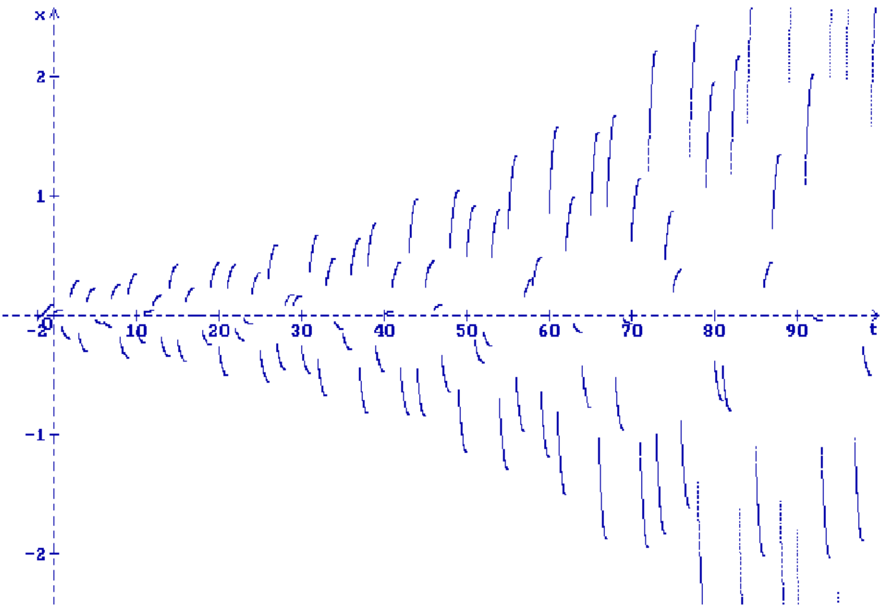


Fig. 9.12 Unstable solution of (9.75) in the point $B(0.54, -2.3)$

9.3 Nonlinear Difference Equation

9.3.1 Nonstationary Systems with Monotone Coefficients

Consider the scalar difference equation

$$
\begin{aligned}
x(t+1) = &-\sum_{j=0}^{[t]+r} a(t,j) f\big(x(t-j)\big) \\
&+ \sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\xi(t+1), \quad t > -1, \\
x(s) = &\ \phi(s), \quad s \in \big[-(r+1), 0\big],
\end{aligned}
\tag{9.90}
$$

where the functions $f(x)$, $g(x)$ and the stochastic process $\xi(t)$ satisfy the conditions

$$
\begin{aligned}
&0 < c_1 \le \frac{f(x)}{x} \le c_2, \quad x \ne 0, \qquad \big|g(x)\big| \le c_3|x|, \\
&\mathbf{E}_t \xi(t+1) = 0, \qquad \mathbf{E}_t \xi^2(t+1) = 1, \quad t > -1.
\end{aligned}
\tag{9.91}
$$

Put

$$
F(t,j,a,\sigma) = \big|a(t,j)\big| \sum_{k=0}^{[t]+r} \big|\sigma(t,k)\big|, \tag{9.92}
$$

$$
G(a,\sigma) = \frac{1}{2} \sup_{t \ge 0} \sum_{j=0}^{\infty} \big(F(t+j,j,a,\sigma) + F(t+j,j,\sigma,a)\big). \tag{9.93}
$$

Theorem 9.5 *Let the coefficients $a(t,j), \sigma(t,j), t > -1, j = 0, \dots, [t]+r$, satisfy the conditions*

$$
a(t,j) \ge a(t,j+1) \ge 0, \tag{9.94}
$$

$$
a(t+1,j+1) - a(t+1,j) - a(t,j) + a(t,j-1) \ge 0, \tag{9.95}
$$

$$
\begin{aligned}
a = &\sup_{t > -1} \big(a(t+1,0) + a(t,0) - a(t+1,1)\big) \\
&< 2\left[\frac{1}{c_2} - \frac{c_3}{c_1}\left(c_3 G(\sigma,\sigma) + \frac{c_2 - c_1}{2} G(a,\sigma)\right)\right]
\end{aligned}
\tag{9.96}
$$

(*here and everywhere below it is supposed that $a(t,-1) = a$ and $a(t,j) = 0$ for $j > [t]+r$*). *Then the trivial solution of* (9.90) *is asymptotically mean square quasistable*.

For proving this theorem we will need in the following lemma.

Lemma 9.2 *Let $\eta(t)$ is a scalar $\mathfrak{F}_t$-measurable process that satisfies the condition*

$$0 < c_1 \le \eta(t) \le c_2 \tag{9.97}$$

and $\xi(t)$ is a $\mathfrak{F}_t$-measurable stationary stochastic process that satisfies condition (9.4). *Then*

$$\left|\mathbf{E}_t\left[\eta(t+1)\xi(t+1)\right]\right| \le \frac{c_2 - c_1}{2}. \tag{9.98}$$

Proof Put $\Omega_t^+ = \{\omega : \xi(t+1,\omega) \ge 0\}$, $\Omega_t^- = \{\omega : \xi(t+1,\omega) < 0\}$. Let $\mathbf{P}_t$ be the measure corresponding to the conditional expectation $\mathbf{E}_t$. Therefore,

$$\begin{aligned}
&\mathbf{E}_t\left[\eta(t+1)\xi(t+1)\right]\\
&\quad= \int_{\Omega} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)\\
&\quad= \int_{\Omega_t^+} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) + \int_{\Omega_t^-} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)\\
&\quad\le c_2 \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) + c_1 \int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega).
\end{aligned}$$

Using (9.91), we have

$$\mathbf{E}_t\xi(t+1) = \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) + \int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) = 0.$$

So,

$$\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) = -\int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega) = \int_{\Omega_t^-} \left|\xi(t+1,\omega)\right|\mathbf{P}_t(\mathrm{d}\omega)$$

and

$$\begin{aligned}
\mathbf{E}_t\left|\xi(t+1)\right| &= \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)\\
&\quad + \int_{\Omega_t^-} \left|\xi(t+1,\omega)\right|\mathbf{P}_t(\mathrm{d}\omega) = 2\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega).
\end{aligned}$$

Therefore, by virtue of (9.91)

$$\begin{aligned}
&\mathbf{E}_t\left[\eta(t+1)\xi(t+1)\right]\\
&\quad\le (c_2 - c_1)\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)\\
&\quad= \frac{c_2 - c_1}{2}\mathbf{E}_t\left|\xi(t+1)\right| \le \frac{c_2 - c_1}{2}\sqrt{\mathbf{E}_t\xi^2(t+1)} = \frac{c_2 - c_1}{2}.
\end{aligned}$$

Similarly

$$
\begin{aligned}
\mathbf{E}_t\big[\eta(t+1)\xi(t+1)\big] &\ge c_1\int_{\Omega_t^+}\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)+c_2\int_{\Omega_t^-}\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)\\
&=(c_1-c_2)\int_{\Omega_t^+}\xi(t+1,\omega)\mathbf{P}_t(\mathrm{d}\omega)=\frac{c_1-c_2}{2}\mathbf{E}_t\big|\xi(t+1)\big|\\
&\ge\frac{c_1-c_2}{2}\sqrt{\mathbf{E}_t\xi^2(t+1)}=\frac{c_1-c_2}{2}.
\end{aligned}
$$

Thus, (9.98) is true. The proof is completed. □

Proof of Theorem 9.5 It is enough to construct for (9.90) Lyapunov functional $V(t)$ that satisfies the conditions of Theorem 9.2. Put $V(t)=V_1(t)+V_2(t)$, where

$$
V_1(t)=x(t)f\big(x(t)\big),\qquad V_2(t)=\sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j}f\big(x(t-k)\big)\right)^2. \tag{9.99}
$$

The nonnegative numbers $\alpha(t,j)$ are defined here in the following way:

$$
\alpha(t,j)=\frac{a(t,j)-a(t,j+1)}{2-ac_2},\quad t>-1,\ j=0,1,\dots,[t]+r. \tag{9.100}
$$

From (9.100), (9.94) and (9.95) it follows that the numbers $\alpha(t,j)$ satisfy the conditions

$$
0\le\alpha(t+1,j)\le\alpha(t,j-1),\quad t>-1,\ j=0,1,\dots,[t]+r. \tag{9.101}
$$

Using (9.99) and (9.91), we have

$$
\begin{aligned}
\Delta V_1(t)&=-x(t)f\big(x(t)\big)+x(t+1)f\big(x(t+1)\big)\\
&\le -c_1x^2(t)+x(t+1)f\big(x(t+1)\big).
\end{aligned}
$$

Using (9.99) and (9.101), we obtain

$$
\begin{aligned}
V_2(t)&=\sum_{j=0}^{[t]+1+r}\alpha(t+1,j)\left(\sum_{k=0}^{j}f\big(x(t+1-k)\big)\right)^2\\
&\quad-\sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j}f\big(x(t-k)\big)\right)^2\\
&=\sum_{j=0}^{[t]+1+r}\big(\alpha(t+1,j)-\alpha(t,j-1)\big)\left(\sum_{k=0}^{j}f\big(x(t+1-k)\big)\right)^2+Q(t)\le Q(t),
\end{aligned}
$$

where

$$Q(t) = \sum_{j=0}^{[t]+1+r} \alpha(t, j-1)\left(\sum_{k=0}^{j} f\big(x(t+1-k)\big)\right)^2 - \sum_{j=0}^{[t]+r} \alpha(t, j)\left(\sum_{k=0}^{j} f\big(x(t-k)\big)\right)^2.$$

Transform $Q(t)$ in the following way:

$$\begin{aligned}
Q(t) &= \alpha(t,-1) f^2\big(x(t+1)\big) - \sum_{j=0}^{[t]+r} \alpha(t, j)\left(\sum_{k=0}^{j} f\big(x(t-k)\big)\right)^2 \\
&\quad + \sum_{j=1}^{[t]+1+r} \alpha(t, j-1)\left(\sum_{k=0}^{j} f\big(x(t+1-k)\big)\right)^2 \\
&= \alpha(t,-1) f^2\big(x(t+1)\big) - \sum_{j=0}^{[t]+r} \alpha(t, j)\left(\sum_{k=0}^{j} f\big(x(t-k)\big)\right)^2 \\
&\quad + \sum_{j=0}^{[t]+r} \alpha(t, j)\left(f\big(x(t+1)\big) + \sum_{k=1}^{j+1} f\big(x(t+1-k)\big)\right)^2 \\
&= \alpha(t,-1) f^2\big(x(t+1)\big) \\
&\quad + \sum_{j=0}^{[t]+r} \alpha(t, j)\left[\left(f\big(x(t+1)\big) + \sum_{k=0}^{j} f\big(x(t-k)\big)\right)^2 \right. \\
&\quad \left. - \left(\sum_{k=0}^{j} f\big(x(t-k)\big)\right)^2\right] \\
&= \alpha(t,-1) f^2\big(x(t+1)\big) \\
&\quad + \sum_{j=0}^{[t]+r} \alpha(t, j)\left[f^2\big(x(t+1)\big) + 2 f\big(x(t+1)\big) \sum_{k=0}^{j} f\big(x(t-k)\big)\right] \\
&= f^2\big(x(t+1)\big) \sum_{j=-1}^{[t]+r} \alpha(t, j) + 2 f\big(x(t+1)\big) \sum_{k=0}^{[t]+r} f\big(x(t-k)\big) \sum_{j=k}^{[t]+r} \alpha(t, j).
\end{aligned}$$

From (9.100) and $a(t,[t]+r+1)=0$ it follows that $\sum_{j=k}^{[t]+r}\alpha(t,j)=(2-ac_2)^{-1}a(t,k)$. Since $a(t,-1)=a$ we have

$$\Delta V_2(t)\le\frac{af^2(x(t+1))}{2-ac_2}+\frac{2f(x(t+1))}{2-ac_2}\sum_{k=0}^{[t]+r}a(t,k)f\big(x(t-k)\big).$$

Therefore using (9.91) and (9.90), for $x(t+1)\neq 0$ we obtain

$$\begin{aligned}\Delta V_2(t)&\le\frac{a}{2-ac_2}\frac{f(x(t+1))}{x(t+1)}x(t+1)f\big(x(t+1)\big)\\&\quad+\frac{2f(x(t+1))}{2-ac_2}\left(\sum_{j=0}^{[t]+r}\sigma(t,j)g\big(x(t-j)\big)\xi(t+1)-x(t+1)\right)\\&\le\frac{ac_2-2}{2-ac_2}x(t+1)f\big(x(t+1)\big)\\&\quad+\frac{2f(x(t+1))}{2-ac_2}\sum_{j=0}^{[t]+r}\sigma(t,j)g\big(x(t-j)\big)\xi(t+1)\\&=-x(t+1)f\big(x(t+1)\big)\\&\quad+\frac{2}{2-ac_2}\sum_{j=0}^{[t]+r}\sigma(t,j)g\big(x(t-j)\big)\xi(t+1)f\big(x(t+1)\big).\end{aligned}$$

As a result for the functional $V(t)$ we have

$$\mathbf{E}\Delta V(t)\le -c_1\mathbf{E}x^2(t)+\frac{2}{2-ac_2}Z(t),\tag{9.102}$$

where

$$Z(t)=\left|\sum_{j=0}^{[t]+r}\sigma(t,j)\mathbf{E}g\big(x(t-j)\big)\xi(t+1)f\big(x(t+1)\big)\right|.$$

Put now $\eta(t)=\frac{f(x(t))}{x(t)}$, $\mathbf{E}_t=\mathbf{E}\{./\mathfrak{F}_t\}$, $t>0$. Then via (9.90) we obtain

$$\begin{aligned}&\mathbf{E}g\big(x(t-j)\big)\xi(t+1)f\big(x(t+1)\big)\\&\quad=\mathbf{E}g\big(x(t-j)\big)\mathbf{E}_t\big[x(t+1)\eta(t+1)\xi(t+1)\big]\\&\quad=\mathbf{E}g\big(x(t-j)\big)\left[-\sum_{k=0}^{[t]+r}a(t,k)f\big(x(t-k)\big)\mathbf{E}_t\big[\eta(t+1)\xi(t+1)\big]\right.\\&\qquad\left.+\sum_{k=0}^{[t]+r}\sigma(t,k)g\big(x(t-k)\big)\mathbf{E}_t\big[\eta(t+1)\xi^2(t+1)\big]\right].\end{aligned}$$

From (9.91) it follows that $\mathbf{E}_t[\eta(t+1)\xi^2(t+1)] \le c_2$ and therefore

$$Z(t) \le \mathbf{E}\left|\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right| \left|\sum_{k=0}^{[t]+r} a(t,k) f\big(x(t-k)\big)\right| \left|\mathbf{E}_t\big[\eta(t+1)\xi(t+1)\big]\right| + c_2 \mathbf{E}\left(\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right)^2.$$

Via Lemma 9.2

$$Z(t) \le c_2 \mathbf{E}\left(\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right)^2 + \frac{c_2 - c_1}{2} \mathbf{E}\left|\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right| \left|\sum_{k=0}^{[t]+r} a(t,k) f\big(x(t-k)\big)\right|. \quad (9.103)$$

Using (9.91) and (9.92), we have

$$\begin{aligned} \mathbf{E}\left(\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right)^2 &\le c_3^2 \mathbf{E}\left(\sum_{j=0}^{[t]+r} \big|\sigma(t,j)\big|\big|x(t-j)\big|\right)^2 \\ &\le c_3^2 \sum_{j=0}^{[t]+r} F(t,j,\sigma,\sigma)\mathbf{E}x^2(t-j) \end{aligned} \quad (9.104)$$

and

$$\begin{aligned} &\mathbf{E}\left|\sum_{j=0}^{[t]+r} \sigma(t,j) g\big(x(t-j)\big)\right| \left|\sum_{k=0}^{[t]+r} a(t,k) f\big(x(t-k)\big)\right| \\ &\le c_2 c_3 \sum_{j=0}^{[t]+r} \sum_{k=0}^{[t]+r} \big|\sigma(t,j)\big|\big|a(t,k)\big| \mathbf{E}\big|x(t-j)\big|\big|x(t-k)\big| \\ &\le \frac{1}{2} c_2 c_3 \left[\sum_{k=0}^{[t]+r} F(t,k,a,\sigma)\mathbf{E}x^2(t-k) + \sum_{j=0}^{[t]+r} F(t,j,\sigma,a)\mathbf{E}x^2(t-j)\right] \\ &= \frac{1}{2} c_2 c_3 \sum_{j=0}^{[t]+r} \big[F(t,j,a,\sigma) + F(t,j,\sigma,a)\big]\mathbf{E}x^2(t-j). \end{aligned} \quad (9.105)$$

Using (9.102)–(9.105), we obtain

$$\mathbf{E}\Delta V(t) \le -c_1\mathbf{E}x^2(t) + \sum_{j=0}^{[t]+r} Q(t,j)\mathbf{E}x^2(t-j),$$

where

$$Q(t,j) = \frac{2c_2c_3}{2-ac_2}\Big[c_3F(t,j,\sigma,\sigma) + \frac{1}{4}(c_2-c_1)\big(F(t,j,a,\sigma)+F(t,j,\sigma,a)\big)\Big]. \tag{9.106}$$

From Theorem 9.2 it follows that the inequality

$$\sum_{j=0}^{\infty} Q(t+j,j) < c_1$$

is a sufficient condition for asymptotic mean square quasistability of the trivial solution of (9.90). Via (9.93), (9.92) and (9.106) this inequality is equivalent to (9.96). The theorem is proven. □

Remark 9.10 As follows from Remark 9.5 by the conditions of Theorem 9.5 the solution of (9.90) is mean square integrable.

Remark 9.11 Suppose that the parameters of (9.90) do not depend on t, i.e. $a(t,j) = a(j)$, $\sigma(t,j) = \sigma(j)$. It is easy to see that in this case $G(a,\sigma) = \hat{a}\hat{\sigma}$, where $\hat{a} = \sum_{j=0}^{\infty}|a(j)|$, $\hat{\sigma} = \sum_{j=0}^{\infty}|\sigma(j)|$. So, the stability conditions (9.94)–(9.96) have the form

$$a(j) \ge a(j+1) \ge 0, \qquad a(j+2) - 2a(j+1) + a(j) \ge 0, \qquad j = 0,1,\dots,$$
$$a(0) - \frac{a(1)}{2} < \frac{1}{c_2} - \frac{c_3}{c_1}\hat{\sigma}\left(c_3\hat{\sigma} + \frac{c_2-c_1}{2}\hat{a}\right).$$

Remark 9.12 Without loss of generality in condition (9.91) we can put $c_3 = 1$ and $c_1 \le c_2 = 1$ or $c_2 \ge c_1 = 1$. In fact, if it is not so we can put for instance $a(t,j)f(x) = \tilde{a}(t,j)\tilde{f}(x)$, where $\tilde{a}(t,j) = c_2a(t,j)$, $\tilde{f}(x) = c_2^{-1}f(x)$. In this case the function $\tilde{f}(x)$ satisfies the condition (9.91) with $c_2 = 1$.

Example 9.4 Consider the equation

$$x(t+1) = \sum_{j=0}^{[t]+r} a(t,j)\big[-\lambda_1 f\big(x(t-j)\big) + \lambda_2 g\big(x(t-j)\big)\xi(t+1)\big], \tag{9.107}$$

where $a(t,j)=\frac{t-j}{(t+2)^2}$, $\lambda_l>0$, $l=1,2$, the functions $f(x)$, $g(x)$ and the stochastic process $\xi(t)$ satisfy the conditions (9.91).

Let us construct the stability condition for this equation using Theorem 9.5. It is easy to check that the conditions (9.94) and (9.95) hold. Calculating a, we have

$$\begin{aligned} a &= \lambda_1 \sup_{t>-1}\left[\frac{t+1}{(t+3)^2}+\frac{t}{(t+2)^2}-\frac{t}{(t+3)^2}\right] \\ &= \lambda_1 \sup_{t>-1}\left[\frac{1}{(t+3)^2}+\frac{t}{(t+2)^2}\right] = A\lambda_1. \end{aligned}$$

Here $A=0.174$ and takes his value in the point $t=1.1295$. Calculating $G(a,\sigma)$, we have

$$\begin{aligned} G(a,\sigma) &= \lambda_1\lambda_2 \sup_{t\geq 0}\left[\sum_{j=0}^{\infty} a(t+j,j)\sum_{k=0}^{[t]+j+r} a(t+j,k)\right] \\ &= \lambda_1\lambda_2 \sup_{t\geq 0}\left[\sum_{j=0}^{\infty}\frac{t}{(t+j+2)^2}\sum_{k=0}^{[t]+j+r}\frac{t+j-k}{(t+j+2)^2}\right] \\ &= \frac{\lambda_1\lambda_2}{2}\sup_{t\geq 0}\left[t\sum_{j=0}^{\infty}\frac{(2t-[t]+j-r)([t]+j+r+1)}{(t+j+2)^4}\right]. \end{aligned}$$

Note that

$$\begin{aligned} \big(2t-[t]+j-r\big)\big([t]+j+r+1\big) &\leq (t+j+1-r)(t+j+1+r) \\ &\leq (t+j+1)^2. \end{aligned}$$

So, via Lemma 1.4

$$G(a,\sigma)\leq \frac{\lambda_1\lambda_2}{2}\sup_{t\geq 0}\left[t\sum_{j=2}^{\infty}\frac{1}{(t+j)^2}\right]\leq \frac{\lambda_1\lambda_2}{2}\sup_{t\geq 0}\left[t\int_{t+1}^{\infty}\frac{ds}{s^2}\right]=\frac{\lambda_1\lambda_2}{2}.$$

Similar $G(\sigma,\sigma)\leq \frac{1}{2}\lambda_2^2$. So, if the condition

$$\frac{\lambda_1}{4}\big[2Ac_1+\lambda_2c_3(c_2-c_1)\big]+\frac{\lambda_2^2c_3^2}{2}<\frac{c_1}{c_2}$$

holds, then the trivial solution of (9.90) is asymptotically mean square quasistable.

9.3.2 Stability in Probability

Consider a quasilinear stochastic Volterra difference equation

$$x(t+\tau)=\sum_{j=0}^{r(t)}a_jx(t-h_j)+\sum_{j=0}^{r(t)}\sigma_jx(t-h_j)\xi(t+\tau)$$
$$+g\big(t,x(t),x(t-h_1),x(t-h_2),\ldots\big),\quad t>t_0-\tau,\quad (9.108)$$

with the initial condition

$$x(\theta)=\phi(\theta),\quad \theta\in\Theta=[t_0-h,t_0],\ h=\tau+\max_{j\geq 1}h_j. \quad (9.109)$$

Here $r(t)=r+[\frac{t-t_0}{\tau}]$, r is a given nonnegative integer, $h_0=0$, $h_j>0$ if $j>0$, a_j, σ_j, $j=0,1,\ldots$ are known constants, the functional g satisfies the condition

$$\big|g(t,x_0,x_1,x_2,\ldots)\big|\leq\sum_{j=0}^{\infty}\gamma_j|x_j|^{\nu_j},\quad \inf_{j\geq 0}\nu_j>1, \quad (9.110)$$

$\phi(\theta)$, $\theta\in\Theta$, is a $\mathfrak{F}_{t_0}$-measurable function, the perturbation $\xi(t)$ is a $\mathfrak{F}_t$-measurable stationary stochastic process with the conditions

$$\mathbf{E}_t\xi(t+\tau)=0,\qquad \mathbf{E}_t\xi^2(t+\tau)=1,\quad t>t_0-\tau. \quad (9.111)$$

A solution of (9.108) and (9.109) is a $\mathfrak{F}_t$-measurable process $x(t)=x(t;t_0,\phi)$, which is equal to the initial function $\phi(t)$ from (9.109) for $t\leq t_0$ and with probability 1 is defined by (9.108) for $t>t_0$.

Definition 9.9 The trivial solution of (9.108) and (9.109) is called stable in probability if for any $\varepsilon>0$, $\varepsilon_1>0$ there exists a $\delta=\delta(\varepsilon,\varepsilon_1,t_0)>0$ such that

$$\mathbf{P}_{t_0}\Big\{\sup_{t\geq t_0}\big|x(t)\big|>\varepsilon\Big\}<\varepsilon_1 \quad (9.112)$$

for any initial function ϕ which is less than δ with probability 1, i.e.,

$$\mathbf{P}\big\{\|\phi\|_0<\delta\big\}=1, \quad (9.113)$$

where $\|\phi\|_0:=\sup_{\theta\in\Theta}|\phi(\theta)|$.

Theorem 9.6 *Let there exist a nonnegative functional* $V(t)=V(t,x(t),x(t-h_1),x(t-h_2),\ldots)$, $t>t_0-\tau$, *and positive numbers* c_0,c_1, *such that*

$$V(t)\geq c_0\big|x(t)\big|^2,\quad t\geq t_0, \quad (9.114)$$

$$V(s)\leq c_1\sup_{\theta\leq s}\big|x(\theta)\big|^2,\quad s\in(t_0-\tau,t_0], \quad (9.115)$$

and

$$\mathbf{E}_t \Delta V(t) \le 0, \quad t > t_0, \tag{9.116}$$

if

$$x(s) \in U_\varepsilon = \{x : |x| \le \varepsilon\}, \quad s \le t. \tag{9.117}$$

Then the trivial solution of (9.108) *and* (9.109) *is stable in probability.*

Proof We will show that for any positive numbers ε and ε_1, there exists a positive number δ such that the solution of (9.108) satisfies (9.112) if the initial condition (9.109) satisfies condition (9.113).

Let $x(t)$, $t \ge t_0$, be a solution of (9.108). Consider the random variable T such that

$$T = \inf\{t \ge t_0 : |x(t)| > \varepsilon\} \tag{9.118}$$

and two events: $\{\sup_{t\ge t_0} |x(t)| > \varepsilon\}$ and $\{|x(T)| \ge \varepsilon\}$. It is easy to see that

$$\Big\{\sup_{t\ge t_0}|x(t)| > \varepsilon\Big\} \subset \big\{|x(T)| \ge \varepsilon\big\}. \tag{9.119}$$

Using $q(T)$, defined in (9.15), from (9.116) and (9.117) we obtain

$$\begin{aligned}\mathbf{E}_{t_0}\sum_{j=1}^{q(T)} \mathbf{E}_{T-j\tau}\Delta V(T-j\tau) &= \sum_{j=1}^{q(T)} \mathbf{E}_{t_0}\big(V\big(T-(j-1)\tau\big) - V(T-j\tau)\big) \\ &= \mathbf{E}_{t_0} V(T) - V(s) \le 0, \end{aligned}\tag{9.120}$$

where $s = T - q(T)\tau \in (t_0 - \tau, t_0]$. So, using (9.119), the Chebyshev inequality, (9.113)–(9.115) and (9.120), we get

$$\begin{aligned}&\mathbf{P}_{t_0}\Big\{\sup_{t\ge t_0}|x(t)| > \varepsilon\Big\} \\ &\quad\le \mathbf{P}_{t_0}\big\{|x(T)| \ge \varepsilon\big\} \\ &\quad\le \frac{\mathbf{E}_{t_0}x^2(T)}{\varepsilon^2} \le \frac{\mathbf{E}_{t_0}V(T)}{c_0\varepsilon^2} \le \frac{V(s)}{c_0\varepsilon^2} \le \frac{c_1\|\phi\|_0^2}{c_0\varepsilon^2} \le \frac{c_1\delta^2}{c_0\varepsilon^2}.\end{aligned} \tag{9.121}$$

Choosing $\delta = \varepsilon\sqrt{\varepsilon_1 c_0/c_1}$, we get (9.112). The proof is completed. □

Remark 9.13 It is easy to see that if $\varepsilon \ge \epsilon_0$, then

$$\mathbf{P}\Big\{\sup_{t\ge t_0}|x(t)| > \varepsilon\Big\} \le \mathbf{P}\Big\{\sup_{t\ge t_0}|x(t)| > \varepsilon_0\Big\}.$$

It means that if the condition (9.112) holds for small enough $\varepsilon > 0$ then it holds for each $\varepsilon > 0$. Thus, for the stability in probability of the trivial solution of (9.108) and (9.109) it is sufficient to prove condition (9.112) for small enough $\varepsilon > 0$.

From Theorem 9.6 it follows that the investigation of the stability in probability of the trivial solution of (9.108) can be reduced to the construction of appropriate Lyapunov functionals.

To get a sufficient condition for stability in probability in terms of the system parameters, denote

$$\alpha_0=\sum_{j=0}^{\infty}|a_j|,\qquad \delta_0=\sum_{j=0}^{\infty}|\sigma_j|,\qquad \gamma=\sum_{j=0}^{\infty}\gamma_j. \tag{9.122}$$

Theorem 9.7 *Let* $\gamma<\infty$ *and*

$$\alpha_0^2+\delta_0^2<1. \tag{9.123}$$

Then the trivial solution of (9.108) *is stable in probability.*

Proof Let us construct a functional $V(t)$ in the form $V(t)=V_1(t)+V_2(t)$, where $V_1(t)=x^2(t)$. Via (9.108) and (9.111) for $t>t_0-\tau$ we have

$$\begin{aligned}
\mathbf{E}_t\Delta V_1(t)&=\mathbf{E}_t x^2(t+\tau)-x^2(t)\\
&=\mathbf{E}_t\Bigg(\sum_{i=0}^{r(t)}a_i x(t-h_i)+g\big(t,x(t),x(t-h_1),x(t-h_2),\ldots\big)\\
&\quad+\sum_{i=0}^{r(t)}\sigma_i x(t-h_i)\xi(t+\tau)\Bigg)^2-x^2(t)\\
&=\Bigg(\sum_{i=0}^{r(t)}a_i x(t-h_i)\Bigg)^2+\Bigg(\sum_{i=0}^{r(t)}\sigma_i x(t-h_i)\Bigg)^2\\
&\quad+g^2\big(t,x(t),x(t-h_1),x(t-h_2),\ldots\big)\\
&\quad+2\sum_{i=0}^{r(t)}a_i x(t-h_i)g\big(t,x(t),x(t-h_1),x(t-h_2),\ldots\big)-x^2(t).
\end{aligned}$$

Via (9.122) and (9.110) we obtain

$$\begin{aligned}
\Bigg(\sum_{i=0}^{r(t)}a_i x(t-h_i)\Bigg)^2&\le\alpha_0\sum_{i=0}^{r(t)}|a_i|x^2(t-h_i),\\
\Bigg(\sum_{i=0}^{r(t)}\sigma_i x(t-h_i)\Bigg)^2&\le\delta_0\sum_{i=0}^{r(t)}|\sigma_i|x^2(t-h_i),\\
g^2\big(t,x(t),x(t-h_1),x(t-h_2),\ldots\big)&\le\gamma\sum_{i=0}^{r(t)}\gamma_i\big|x(t-h_i)\big|^{2\nu_i},
\end{aligned}\tag{9.124}$$

and

$$
\begin{aligned}
&2\sum_{i=0}^{r(t)} a_i x(t-h_i) g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) \\
&\quad \le 2\sum_{i=0}^{r(t)} |a_i| \big|x(t-h_i)\big| \sum_{k=0}^{r(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k} \\
&\quad \le \sum_{i=0}^{r(t)} |a_i| \sum_{k=0}^{r(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} \big(\big|x(t-h_k)\big|^2 + \big|x(t-h_i)\big|^2\big) \\
&\quad \le \alpha_0 \sum_{k=0}^{r(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} \big|x(t-h_k)\big|^2 \\
&\qquad + \sum_{k=0}^{r(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} \sum_{i=0}^{r(t)} |a_i| \big|x(t-h_i)\big|^2.
\end{aligned}
$$

Assume that $x(s) \in U_\varepsilon$ for $s \le t$ and put

$$
\mu_k(\varepsilon) = \sum_{i=0}^{\infty} \gamma_i \varepsilon^{k(\nu_i-1)}, \quad k = 1, 2. \tag{9.125}
$$

If $\varepsilon \le 1$ then $\mu_k(\varepsilon) \le \gamma < \infty$. So,

$$
g^2\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) \le \gamma \sum_{i=0}^{r(t)} \gamma_i \varepsilon^{2(\nu_i-1)} x^2(t-h_i),
$$

$$
\begin{aligned}
&2\sum_{i=0}^{r(t)} a_i x(t-h_i) g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) \\
&\quad \le \sum_{i=0}^{r(t)} \big(\alpha_0 \gamma_i \varepsilon^{\nu_i-1} + \mu_1(\varepsilon)|a_i|\big) x^2(t-h_i).
\end{aligned}
$$

As a result we obtain

$$
\mathbf{E}_t \Delta V_1(t) \le (A_0 - 1)x^2(t) + \sum_{i=1}^{r(t)} A_i x^2(t-h_i),
$$

where

$$
A_i = \big(\alpha_0 + \mu_1(\varepsilon)\big)|a_i| + \delta|\sigma_i| + \big(\alpha_0 + \gamma\varepsilon^{\nu_i-1}\big)\gamma_i \varepsilon^{\nu_i-1}, \quad i = 0, 1, \ldots.
$$

Choosing now the functional V_2 in the form

$$V_2(t)=\sum_{i=1}^{r(t)}\left(x^2(t-h_i)\sum_{l=i}^{\infty}A_l\right), \tag{9.126}$$

we obtain

$$\begin{aligned}\Delta V_2(t)&=\sum_{i=1}^{r(t)+1}\left(x^2(t+\tau-h_i)\sum_{l=i}^{\infty}A_l\right)-\sum_{i=1}^{r(t)}\left(x^2(t-h_i)\sum_{l=i}^{\infty}A_l\right)\\&=\sum_{i=0}^{r(t)}\left(x^2(t-h_i)\sum_{l=i+1}^{\infty}A_l\right)-\sum_{i=1}^{r(t)}\left(x^2(t-h_i)\sum_{l=i}^{\infty}A_l\right)\\&=x^2(t)\sum_{i=1}^{\infty}A_i-\sum_{i=1}^{r(t)}A_ix^2(t-h_i).\end{aligned} \tag{9.127}$$

So, for the functional $V(t)=V_1(t)+V_2(t)$ we have

$$\mathbf{E}_t\Delta V(t)\le\left(\alpha_0^2+\delta_0^2+2\alpha_0\mu_1(\varepsilon)+\gamma\mu_2(\varepsilon)-1\right)x^2(t),\quad t>t_0.$$

From (9.123) it follows that $\mathbf{E}_t\Delta V(t)\le 0$ for small enough ε. It is easy to see that the functional $V(t)$ satisfies the conditions of Theorem 9.6. Therefore, using Remark 9.13, we see that the trivial solution of (9.108) is stable in probability. The proof is completed. □

To get another sufficient condition for stability in probability of the trivial solution of (9.108) put

$$\alpha=\sum_{i=1}^{\infty}\left|\sum_{j=i}^{\infty}a_j\right|,\qquad \beta=\sum_{j=0}^{\infty}a_j. \tag{9.128}$$

Theorem 9.8 *Let $\gamma<\infty$ and*

$$\beta^2+2\alpha|1-\beta|+\delta_0^2<1, \tag{9.129}$$

where γ and δ_0 are defined in (9.121). Then the trivial solution of (9.108) is stable in probability.

Proof It is easy to see that (9.108) can be represented in the form

$$\begin{aligned}x(t+\tau)&=\beta x(t)+\Delta F(t)+\sum_{i=0}^{r(t)}\sigma_i x(t-h_i)\xi(t+\tau)\\&\quad+g\left(t,x(t),x(t-h_1),x(t-h_2),\ldots\right),\end{aligned} \tag{9.130}$$

where

$$F(t) = -\sum_{l=1}^{r(t)} x(t-h_l) \sum_{i=l}^{\infty} a_i, \quad \Delta F(t) = F(t+\tau) - F(t). \tag{9.131}$$

Following the procedure of the construction of the Lyapunov functionals we will construct the Lyapunov functional $V(t)$ in the form $V(t) = V_1(t) + V_2(t)$ again but now with

$$V_1(t) = \big(x(t) - F(t)\big)^2.$$

Calculating $\mathbf{E}_t \Delta V_1(t)$ via (9.111) we get

$$\begin{aligned}
\mathbf{E}_t \Delta V_1(t) &= \mathbf{E}_t \big(x(t+\tau) - F(t+\tau)\big)^2 - \big(x(t) - F(t)\big)^2 \\
&= \mathbf{E}_t \Bigg(\beta x(t) - F(t) + \sum_{i=0}^{r(t)} \sigma_i x(t-h_i) \xi(t+\tau) \\
&\quad + g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big)\Bigg)^2 - \big(x(t) - F(t)\big)^2 \\
&= \big(\beta^2 - 1\big) x^2(t) + \left(\sum_{i=0}^{r(t)} \sigma_i x(t-h_i)\right)^2 + 2(1-\beta) x(t) F(t) \\
&\quad + g^2\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) \\
&\quad + 2\beta x(t) g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big) \\
&\quad - 2F(t) g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big).
\end{aligned}$$

Using the assumption $x(s) \in U_\varepsilon = \{x : |x| \le \varepsilon\}$ for $s \le t$, (9.124) and (9.125) and

$$2\big|x(t)F(t)\big| \le \sum_{l=1}^{r(t)} \left|\sum_{i=l}^{\infty} a_i\right| \big(x^2(t) + x^2(t-h_l)\big) \le \alpha x^2(t) + \sum_{l=1}^{r(t)} \left|\sum_{i=l}^{\infty} a_i\right| x^2(t-h_l),$$

$$\begin{aligned}
&2\big|x(t) g\big(t, x(t), x(t-h_1), x(t-h_2), \ldots\big)\big| \\
&\quad \le \sum_{i=0}^{r(t)} \gamma_i \big|x(t-h_i)\big|^{\nu_i - 1} \big(x^2(t) + x^2(t-h_i)\big) \\
&\quad \le \big(\mu_1(\varepsilon) + \gamma_0 \varepsilon^{\nu_0 - 1}\big) x^2(t) + \sum_{l=1}^{r(t)} \gamma_l \varepsilon^{\nu_l - 1} x^2(t-h_l),
\end{aligned}$$

$$
\begin{aligned}
&2\bigl|F(t)g\bigl(t, x(t), x(t-h_1), x(t-h_2), \dots\bigr)\bigr| \\
&\quad \le \sum_{i=0}^{r(t)} \gamma_i \sum_{l=1}^{r(t)} \left|\sum_{j=l}^{\infty} a_j\right| \varepsilon^{\nu_i - 1}\bigl(x^2(t-h_l) + x^2(t-h_i)\bigr) \\
&\quad \le \alpha\gamma_0\varepsilon^{\nu_0-1}x^2(t) + \sum_{l=1}^{r(t)}\left(\mu_1(\varepsilon)\left|\sum_{j=l}^{\infty} a_j\right| + \alpha\gamma_l\varepsilon^{\nu_l-1}\right)x^2(t-h_l),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathbf{E}_t \Delta V_1(t) \le \bigl(&\beta^2 - 1 + \delta_0|\sigma_0| + \alpha|1-\beta| + \gamma\gamma_0\varepsilon^{2(\nu_0-1)} + |\beta|\mu_1(\varepsilon) \\
&+ \bigl(\alpha + |\beta|\bigr)\gamma_0\varepsilon^{\nu_0-1}\bigr)x^2(t) + \sum_{i=1}^{r(t)} A_i x^2(t-h_i),
\end{aligned} \tag{9.132}
$$

with

$$
A_i = \delta_0|\sigma_i| + \gamma\gamma_i\varepsilon^{2(\nu_i-1)} + \bigl(\mu_1(\varepsilon) + |1-\beta|\bigr)\left|\sum_{j=i}^{\infty} a_j\right| + \bigl(\alpha + |\beta|\bigr)\gamma_i\varepsilon^{\nu_i-1}. \tag{9.133}
$$

Define now the functional V_2 by (9.126) and (9.133). Then using (9.127) and (9.132) we get

$$
\mathbf{E}_t \Delta V(t) \le \bigl(\beta^2 - 1 + 2\alpha|1-\beta| + \delta_0^2 + 2\bigl(\alpha + |\beta|\bigr)\mu_1(\varepsilon) + \gamma\mu_2(\varepsilon)\bigr)x^2(t).
$$

Here $t > t_0 - \tau$, $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are defined in (9.125).

From this and (9.129) it follows that for small enough ε there exists $c_2 > 0$ such that

$$
\mathbf{E}_t \Delta V(t) \le -c_2 x^2(t), \quad t > t_0 - \tau. \tag{9.134}
$$

So, via (9.134) the functional $V(t)$ satisfies condition (9.116). It is easy to check that the functional $V(t)$ satisfies also condition (9.115). But it does not satisfy condition (9.114). So, we cannot use here Theorem 9.6 and have to find another way.

Consider the random variable T that is described by (9.118). From (9.121) we have

$$
\mathbf{P}_{t_0}\Bigl\{\sup_{t \ge t_0}\bigl|x(t)\bigr| > \varepsilon\Bigr\} \le \mathbf{P}_{t_0}\bigl\{\bigl|x(T)\bigr| \ge \varepsilon\bigr\} \le \frac{\mathbf{E}_{t_0}x^2(T)}{\varepsilon^2}. \tag{9.135}
$$

To estimate $\mathbf{E}_{t_0}x^2(T)$ note that

$$
V(T) \ge \bigl(x(T) - F(T)\bigr)^2 \ge x^2(T) - 2x(T)F(T). \tag{9.136}
$$

Besides working via (9.131), (9.128) and (9.15) we may use

$$
\begin{aligned}
2x(T)F(T) &\le \sum_{l=1}^{r(T)}\left|\sum_{j=l}^{\infty}a_j\right|\big(x^2(T)+x^2(T-h_l)\big)\\
&\le \alpha x^2(T)+\sum_{l=1}^{q(T)-1}\left|\sum_{j=l}^{\infty}a_j\right|x^2(T-h_l)+\sum_{l=q(T)}^{r(T)}\left|\sum_{j=l}^{\infty}a_j\right|x^2(T-h_l).
\end{aligned}
$$

Since $T-h_l\le t_0$ for $l\ge q(T)$ then via (9.113) we obtain

$$
2x(T)F(T)\le \alpha x^2(T)+\sum_{l=1}^{q(T)-1}\left|\sum_{j=l}^{\infty}a_j\right|x^2(T-h_l)+\alpha\delta^2. \tag{9.137}
$$

From (9.129) it follows that $\alpha<1$. So, substituting (9.137) into (9.136) we obtain

$$
\mathbf{E}_{t_0}x^2(T)\le(1-\alpha)^{-1}\left[\mathbf{E}_{t_0}V(T)+\sum_{l=1}^{q(T)-1}\left|\sum_{j=l}^{\infty}a_j\right|\mathbf{E}_{t_0}x^2(T-h_l)+\alpha\delta^2\right]. \tag{9.138}
$$

Note that $t_0<T-h_l<T$ for $1\le l<q(T)$. So, from (9.134) it follows that $c_2\mathbf{E}_{t_0}x^2(T-h_l)\le\mathbf{E}_{t_0}V(T-h_l)$. Besides, similar to (9.120) one can show that $\mathbf{E}_{t_0}V(T-h_l)\le V(s)$, where $0\le l<q(T)$, $s=T-q(T)\tau\le t_0$. Hence, via (9.138), (9.115), (9.128) and (9.113) we have

$$
\begin{aligned}
\mathbf{E}_{t_0}x^2(T) &\le (1-\alpha)^{-1}\left[V(s)+\sum_{l=1}^{q(T)}\left|\sum_{j=l}^{\infty}a_j\right|\frac{V(s)}{c_2}+\alpha\delta^2\right]\\
&\le (1-\alpha)^{-1}\left[c_1\|\phi\|_0^2+\sum_{l=1}^{q(T)}\left|\sum_{j=l}^{\infty}a_j\right|\frac{c_1\|\phi\|_0^2}{c_2}+\alpha\delta^2\right]\\
&\le (1-\alpha)^{-1}\left[c_1\left(1+\frac{\alpha}{c_2}\right)+\alpha\right]\delta^2:=C\delta^2.
\end{aligned}
$$

From this and (9.135) by $\delta=\varepsilon\sqrt{\varepsilon_1/C}$, (9.112) follows. Therefore, the trivial solution of (9.108) is stable in probability. The proof is completed. □

Remark 9.14 Note that condition (9.129) can be transformed to the form

$$
\delta_0^2<(1-\beta)(1+\beta-2\alpha),\quad |\beta|<1.
$$

Remark 9.15 As follows from (9.57) and (9.61) inequalities (9.123) and (9.129) are sufficient conditions for asymptotic mean square stability of the trivial solution of (9.108) in the case $g\equiv 0$. It means that the investigation of the stability in probability of nonlinear stochastic difference equations with continuous variable and the order of nonlinearity higher than one can be reduced to the investigation of asymptotic mean square stability of the linear part of this equation.

Remark 9.16 It is easy to see that Theorems 9.6, 9.7 and 9.8 without essential changes can be formulated also for the case $x \in \mathbf{R}^n$.

Example 9.5 Consider the nonlinear difference equation

$$x(t+1) = \sum_{i=0}^{r(t)} 2^{-(i+1)}\big[(-1)^i a + b\xi(t+1)\big]x(t-i) + \gamma \sum_{i=0}^{r(t)} 2^{-(i+1)} x^{\nu_i}(t-i), \tag{9.139}$$

where $r(t) = r + [t - t_0]$, r is a given nonnegative integer, $t > t_0 - 1$, $\nu_i > 1$, $i = 0, 1, \dots$.

From Theorem 9.5 it follows that if

$$a^2 + b^2 < 1, \tag{9.140}$$

then the trivial solution of (9.139) is stable in probability.

To use Theorem 9.8 note that $\alpha = 3^{-1}|a|$, $\beta = 3^{-1}a$. Via Remark 9.14 we find a sufficient condition for stability in probability in the form

$$|b| < \begin{cases} 1 - 3^{-1}a & \text{if } a \in [0, 3), \\ \sqrt{(1 - 3^{-1}a)(1 + a)} & \text{if } a \in (-1, 0). \end{cases} \tag{9.141}$$

In Fig. 9.13 the stability regions are shown obtained by condition (9.140) (number 1) and condition (9.141) (number 2). One can see that both these regions supplement each other.

Example 9.6 Consider the nonlinear difference equation

$$x(t+1) = \frac{ax(t)}{1 + \gamma \sin(x(t - r(t)))} + bx(t-k)\xi(t+1), \quad t > t_0 - 1, \tag{9.142}$$

where $r(t) = r + [t - t_0]$, r is a given nonnegative integer, $k \ge 0$ and

$$|\gamma| < 1. \tag{9.143}$$

Via Remark 9.4 the inequality

$$\frac{a^2}{(1 - |\gamma|)^2} + b^2 < 1 \tag{9.144}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (9.142).

To get a sufficient condition for stability in probability of the trivial solution of equation (9.142) rewrite it in the form

$$x(t+1) = ax(t) + g\big(x(t), x\big(t - r(t)\tau\big)\big) + bx(t-k)\xi(t+1), \tag{9.145}$$

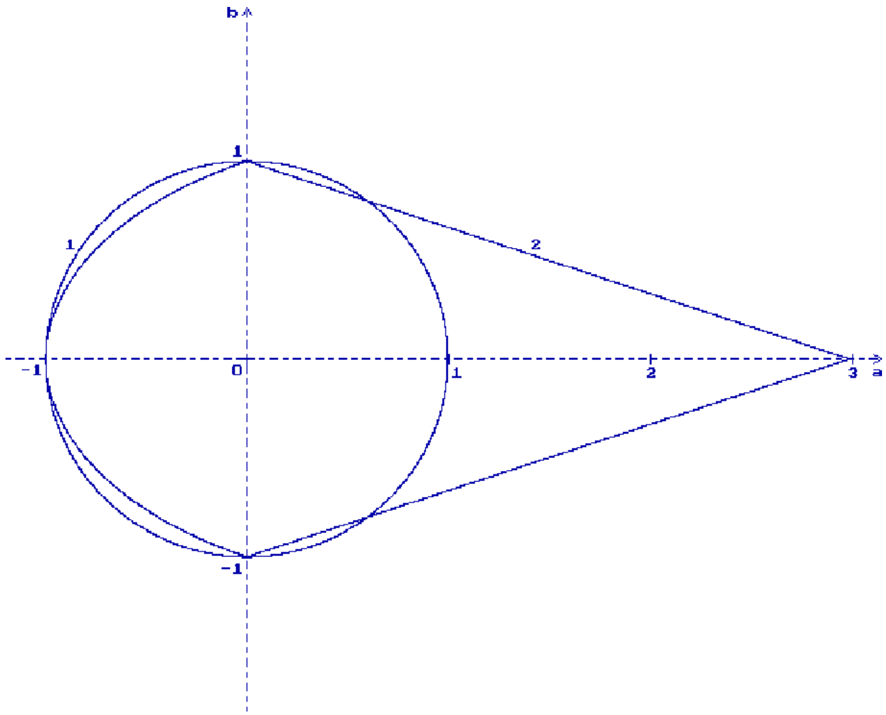


Fig. 9.13 Stability regions for (9.139)

where

$$g(x, y) = -\frac{a\gamma x \sin(y)}{1 + \gamma \sin(y)}.$$

Using the inequality $|\sin(y)| \le |y|$ in the numerator and the conditions (9.143), $|\sin(y)| \le 1$ in the denominator we have

$$\left|g(x, y)\right| \le \frac{|a\gamma x y|}{1 - |\gamma|} \le \frac{|a\gamma|}{2(1 - |\gamma|)}\left(x^2 + y^2\right).$$

It means that the function $g(x, y)$ satisfies condition (9.110). Therefore, the condition

$$a^2 + b^2 < 1,$$

which can be obtained from (9.144) by $\gamma = 0$, is a sufficient condition for asymptotic mean square stability of the linear part of (9.145) and as follows from Theorem 9.7 is a sufficient condition for the stability in probability of the trivial solution of (9.142) for all γ satisfying the condition (9.143).

9.4 Volterra Equations of Second Type

Consider the stochastic difference equation

$$x(t+\tau)=\eta(t+\tau)+F\big(t,x(t),x(t-h_1),x(t-h_2),\dots\big),\quad t>t_0-\tau,\quad(9.146)$$

with the initial condition

$$x(\theta)=\phi(\theta),\quad \theta\in\Theta=[t_0-h,t_0],\ h=\tau+\max_{j\ge1}h_j.\quad(9.147)$$

Here $\eta\in H$, H is a space of $\mathfrak{F}_t$-measurable functions $x(t)\in R^n$, $t\ge t_0$, with norms

$$\|x\|^2=\sup_{t\ge t_0}\mathbf{E}\big|x(t)\big|^2,\qquad \|x\|_1^2=\sup_{t\in[t_0,t_0+\tau]}\mathbf{E}\big|x(t)\big|^2.$$

$h_0,h_1,\dots$ are positive constants, $\phi(\theta)$, $\theta\in\Theta$, is a $\mathfrak{F}_{t_0}$-measurable function such that

$$\|\phi\|_0^2=\sup_{\theta\in\Theta}\mathbf{E}\big|\phi(\theta)\big|^2<\infty,$$

the functional $F\in\mathbf{R}^n$ satisfies the condition

$$\big|F(t,x_0,x_1,x_2,\dots)\big|^2\le\sum_{j=0}^{\infty}a_j|x_j|^2,\qquad A=\sum_{j=0}^{\infty}a_j<\infty.\quad(9.148)$$

A solution of (9.146) and (9.147) is a $\mathfrak{F}_t$-measurable process $x(t)=x(t;t_0,\phi)$, which is equal to the initial function $\phi(t)$ from (9.147) for $t\le t_0$ and with probability 1 is defined by (9.146) for $t>t_0$.

Together with (9.146) we will consider the auxiliary difference equation

$$x(t+\tau)=F\big(t,x(t),x(t-h_1),x(t-h_2),\dots\big),\quad t>t_0-\tau,\quad(9.149)$$

with initial condition (9.147) and the functional F, satisfying condition (9.148).

Theorem 9.9 *Let the process $\eta(t)$ satisfy the condition $\|\eta\|_1^2<\infty$ and let there exist a nonnegative functional $V(t)=V(t,x(t),x(t-h_1),x(t-h_2),\dots)$, positive numbers c_1, c_2 and nonnegative function $\gamma(t)$, $t>t_0-\tau$, such that*

$$\hat\gamma=\sup_{s\in(t_0-\tau,t_0]}\sum_{j=0}^{\infty}\gamma(s+j\tau)<\infty,\quad(9.150)$$

$$\mathbf{E}V(s)\le c_1\sup_{\theta\le s}\mathbf{E}\big|x(\theta)\big|^2,\quad s\in(t_0-\tau,t_0],\quad(9.151)$$

$$\mathbf{E}\Delta V(t)\le -c_2\mathbf{E}\big|x(t)\big|^2+\gamma(t),\quad t\ge t_0;\quad(9.152)$$

$\Delta V(t)$ is defined in (9.10). Then the solution of (9.146) and (9.147) is uniformly mean square summable.

Proof Rewrite condition (9.152) in the form

$$\mathbf{E}\Delta V(t+j\tau) \le -c_2\mathbf{E}\big|x(t+j\tau)\big|^2 + \gamma(t+j\tau), \quad t \ge t_0,\ j = 0,1,\dots.$$

Summing this inequality from $j=0$ to $j=i$, by virtue of (9.10) we obtain

$$\mathbf{E}V\big(t+(i+1)\tau\big) - \mathbf{E}V(t) \le -c_2\sum_{j=0}^{i}\mathbf{E}\big|x(t+j\tau)\big|^2 + \sum_{j=0}^{i}\gamma(t+j\tau).$$

Therefore,

$$c_2\sum_{j=0}^{\infty}\mathbf{E}\big|x(t+j\tau)\big|^2 \le \mathbf{E}V(t) + \sum_{j=0}^{\infty}\gamma(t+j\tau), \quad t \ge t_0. \tag{9.153}$$

Let us show that the right-hand side of inequality (9.153) is bounded. In fact, via (9.152) and (9.10), similarly to (9.14) we obtain

$$\begin{aligned}
\mathbf{E}V(t) &\le \mathbf{E}V(t-\tau) + \gamma(t-\tau) \\
&\le \mathbf{E}V(t-2\tau) + \gamma(t-2\tau) + \gamma(t-\tau) \le \cdots \\
&\le \mathbf{E}V(t-i\tau) + \sum_{j=1}^{i}\gamma(t-j\tau) \le \cdots \le \mathbf{E}V(s) + \sum_{j=1}^{q(t)}\gamma(t-j\tau), \\
&t \ge t_0,\ s = t - q(t)\tau \in (t_0-\tau, t_0],
\end{aligned} \tag{9.154}$$

where $q(t)$ is defined in (9.15).

Since $t = s + q(t)\tau$ we have

$$\begin{aligned}
\sum_{j=0}^{\infty}\gamma(t+j\tau) &= \sum_{j=0}^{\infty}\gamma\big(s+\big(q(t)+j\big)\tau\big) = \sum_{j=q(t)}^{\infty}\gamma(s+j\tau), \\
\sum_{j=1}^{q(t)}\gamma(t-j\tau) &= \sum_{j=1}^{q(t)}\gamma\big(s+\big(q(t)-j\big)\tau\big) = \sum_{j=0}^{q(t)-1}\gamma(s+j\tau).
\end{aligned} \tag{9.155}$$

So, via (9.153), (9.154), (9.155), (9.150) and (9.151), we obtain

$$c_2\sum_{j=0}^{\infty}\mathbf{E}\big|x(t+j\tau)\big|^2 \le c_1\|\phi\|_0^2 + \hat{\gamma}, \tag{9.156}$$

i.e. the solution of (9.146) and (9.147) is uniformly mean square summable. The theorem is proven. □

Remark 9.17 Similarly to Remark 9.5 one can show that if the condition (9.150) is replaced by the condition

$$\int_{t_0}^{\infty} \gamma(t)\,\mathrm{d}t < \infty$$

then the solution of (9.146) for each initial function (9.147) is mean square integrable.

Remark 9.18 Suppose that for (9.149) the conditions of Theorem 9.9 hold with $\gamma(t) \equiv 0$. Then the trivial solution of (9.149) is asymptotically mean square quasistable. In fact, in the case $\gamma(t) \equiv 0$ from inequality (9.156) for (9.149) ($\eta(t) \equiv 0$) it follows that $c_2 \mathbf{E}|x(t)|^2 \le c_1 \|\phi\|_0^2$ and $\lim_{j\to\infty} \mathbf{E}|x(t+j\tau)|^2 = 0$. It means that the trivial solution of (9.149) is asymptotically mean square quasistable.

The formal procedure of the construction of the Lyapunov functionals that was described for (9.1) can be used here by the assumption $a_2 \equiv 0$. Moreover, by virtue of Theorem 9.9 some results obtained above can be immediately applied for the equations of type (9.146).

Consider for example the equation

$$x(t+1) = \eta(t+1) + \sum_{j=0}^{[t]+r} a_j x(t-j),$$

$$t > -1,\ x(s) = \phi(s),\ s \in \big[-(r+1), 0\big], \tag{9.157}$$

and the inequality

$$\alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k < d_{k+1,k+1}^{-1}, \tag{9.158}$$

where α_{k+1}, β_k and $d_{k+1,k+1}$ are defined by (9.44), (9.48) and the matrix equation (9.40).

Inequality (9.158) follows from (9.53) by $\delta_0 = 0$. So, similarly to (9.53) by virtue of Theorem 9.9 it can be proven that if the inequality (9.158) holds then the solution of (9.157) is uniformly mean square summable.

Consider also the inequality

$$\beta^2 + 2\alpha|1-\beta| < 1 \tag{9.159}$$

where β and α are defined in (9.58) and (9.60). Condition (9.159) is a particular case of (9.61) by $\delta_0 = 0$. If inequality (9.159) holds then the solution of (9.157) is uniformly mean square summable.

Example 9.7 Consider the difference equation

$$x(t+1) = \eta(t+1) + ax(t) + \sum_{j=1}^{[t]+r} b^j x(t-j), \quad t > -1,$$
$$x(\theta) = \phi(\theta), \quad \theta \in \big[-(r+1), 0\big],\ r \ge 0. \tag{9.160}$$

Each of the conditions (9.76)–(9.80) by $\sigma = 0$ is a sufficient condition for mean square summability of the solution of (9.160).

For example, for the conditions (9.76), (9.77) and (9.80) we have respectively

$$|a| < \frac{1-2|b|}{1-|b|}, \quad |b| < \frac{1}{2},$$

$$|a| < \left(\frac{1-2|b|}{1-|b|}\right)\left(\frac{1-b}{1-|b|}\right), \quad |b| < \frac{1}{2},$$

and

$$-\frac{1-3|b|}{(1-b)(1-|b|)} < a < \frac{1-2b}{1-b}, \quad |b| < 1.$$

The bounds of summability regions for the solution of (9.160) given by the conditions (9.76)–(9.80) are shown in Fig. 9.6 (curves 1–5).

Chapter 10
Difference Equations as Difference Analogues of Differential Equations

Functional differential equations arise in the modeling of hereditary systems such as ecological and biological systems, chemical and mechanical systems and many, many other systems. The long-term behavior and stability of such systems is an important area for investigation. For example, will a population decline to dangerously low levels? Could a small change in the environmental conditions have drastic consequences on the long-term survival of the population? There is a growing body of works devoted to such type of investigations (see, e.g., [3, 4, 16, 20–24, 26, 28–31, 34–36, 41, 42, 49, 56, 57, 63, 74–77, 83, 84, 92, 93, 95, 98, 101, 102, 105–107, 110, 152, 154, 159, 160, 163, 164, 167, 168, 170–173, 175, 183, 185, 188, 190–192, 202, 216, 218, 235, 242, 246, 247, 252, 254, 255, 263–269, 272, 275, 277]).

Analytical solutions of functional differential equations are generally unavailable and a lot of different numerical methods have been adopted for obtaining approximate solutions [15, 32, 33, 61, 79, 81, 103, 104, 113, 161, 165, 166, 187, 213, 217, 219–225, 248, 249, 252, 259]. A natural question to ask is: do the numerical solutions preserve the stability properties of the exact solution? Thus, to use numerical investigation of functional differential equations it is very important to know if the considered difference analogue of the original differential equation is reliable to preserve some general properties of this equation, in particular, the property of stability.

In this chapter the capability of difference analogues of differential equations to save the property of stability of solutions of the differential equations considered is studied. In particular, sufficient conditions for the step of discretization, at which stability of solution is saved, are obtained for some known mathematical models.

10.1 Stability Conditions for Stochastic Differential Equations

To begin with, let us consider some simple examples with the possibility to get stability conditions for stochastic differential equations using its difference analogues.

Example 10.1 Consider the scalar stochastic differential equation of neutral type

$$\dot{x}(t) + ax(t) + bx(t-h) + c\dot{x}(t-h) + \sigma x(t-\tau)\dot{w}(t) = 0, \quad t \geq 0, \qquad (10.1)$$

DOI 10.1007/978-0-85729-685-6_10,

and its difference analogue via the Euler–Maruyama scheme [184]

$$x_{i+1} = (1 - \Delta a)x_i - cx_{i-k+1} + (c - \Delta b)x_{i-k} + \sigma\sqrt{\Delta}x_{i-m}\xi_{i+1}, \quad i = 0, 1, \ldots. \tag{10.2}$$

Here $w(t)$ is the standard Wiener process,

$$\begin{aligned} &\Delta > 0, \qquad t_i = i\Delta, \qquad x_i = x(t_i), \qquad k = \frac{h}{\Delta}, \qquad m = \frac{\tau}{\Delta}, \\ &\xi_{i+1} = \frac{w(t_{i+1}) - w(t_i)}{\sqrt{\Delta}}, \qquad \mathbf{E}\xi_i = 0, \qquad \mathbf{E}\xi_i^2 = 1, \quad i = 0, 1, \ldots, \end{aligned} \tag{10.3}$$

and it is supposed that k and m are integers.

Using (3.2) and (3.4) and stability condition (3.5) for (10.2) we have

$$\alpha = |b|\Delta(k-1) + |c - \Delta b| = |b|(h - \Delta) + |c - \Delta b|, \qquad \beta = 1 - \Delta(a + b),$$

and we have a sufficient condition for asymptotic mean square stability of the trivial solution of (10.2) in the form

$$\big(1 - \Delta(a+b)\big)^2 + 2\Delta|a+b|\big(|b|(h-\Delta) + |c - \Delta b|\big) + \Delta\sigma^2 < 1$$

or

$$\Delta(a+b)^2 + 2|a+b|\big(|b|(h-\Delta) + |c-\Delta b|\big) + \sigma^2 < 2(a+b). \tag{10.4}$$

Let $\Delta \to 0$. Then from (10.4) the known [136] sufficient condition for asymptotic mean square stability of the trivial solution of (10.1) follows:

$$(a+b)\big(1 - |b|h - |c|\big) > \frac{\sigma^2}{2}, \qquad |b|h + |c| < 1. \tag{10.5}$$

Example 10.2 Consider the scalar stochastic integro-differential equation

$$\dot{x}(t) = ax(t) + b\int_{t-h}^{t} x(s)\,\mathrm{d}s + \sigma x(t-\tau)\dot{w}(t), \tag{10.6}$$

where $w(t)$ is the standard Wiener process. Using (10.3), the Euler–Maruyama scheme and θ-method ($\theta \in [0, 1]$) for a difference representation of the integral consider a difference analogue of (10.6) in the form

$$x_{i+1} = \big[1 + \Delta a + \Delta^2 b(1-\theta)\big]x_i + \Delta^2 b\left(\sum_{j=1}^{k-1} x_{i-j} + \theta x_{i-k}\right) + \sigma\sqrt{\Delta}x_{i-m}\xi_{i+1}. \tag{10.7}$$

Using (3.2) and (3.4) for (10.7) we have

$$\alpha = |b|h\Delta\left(\theta + \frac{k-1}{2}\right) = \frac{1}{2}|b|h\big(\Delta(2\theta - 1) + h\big), \qquad \beta = 1 + \Delta(a + bh).$$

So, the sufficient condition (3.5) for asymptotic mean square stability of the trivial solution of (10.7) takes the form

$$\big(1+\Delta(a+bh)\big)^2+\Delta h|b||a+bh|\big(\Delta(2\theta-1)+h\big)+\Delta\sigma^2<1$$

or

$$\Delta(a+bh)^2+h|b||a+bh|\big(\Delta(2\theta-1)+h\big)+\sigma^2<2|a+bh|. \tag{10.8}$$

Let $\Delta\to 0$. Then from (10.8) the known [134] sufficient condition for asymptotic mean square stability of the trivial solution of (10.6) follows:

$$|a+bh|\left(1-|b|\frac{h^2}{2}\right)>\frac{\sigma^2}{2},\quad a+bh<0. \tag{10.9}$$

Remark 10.1 One can see that if condition (10.5) (or (10.9)) holds, then inequality (10.4) (or (10.8)) at the same time is a sufficient condition on the step of discretization Δ by which difference analogue (10.2) (or (10.7)) saves the stability property of the solution of the initial equation (10.1) (or (10.6)).

10.2 Difference Analogue of the Mathematical Model of the Controlled Inverted Pendulum

10.2.1 Mathematical Model of the Controlled Inverted Pendulum

The problem of stabilization for the controlled inverted pendulum during many years is very popular among researchers (see, for instance [3, 4, 26, 28, 29, 107, 110, 163, 164, 185, 188, 192, 218, 242, 247, 252]). The linearized mathematical model of the controlled inverted pendulum can be described by the linear differential equation of the second order

$$\ddot{x}(t)-ax(t)=u(t),\quad a>0,\ t\geq 0. \tag{10.10}$$

The classical way of stabilization [110] uses the control $u(t)=-b_1x(t)-b_2\dot{x}(t)$, $b_1>a$, $b_2>0$. But this type of control, which represents instantaneous feedback, is quite difficult to realize because usually it is necessary to have some finite time to make measurements of the coordinates and velocities, to treat the results of the measurements and to implement them in the control action.

In [28, 29, 242, 247] the control $u(t)$ is proposed that does not depend on the velocity but it depends on the previous values of the trajectory $x(s)$, $s\leq t$, and has the form

$$u(t)=\int_0^\infty \mathrm{d}K(\tau)\,x(t-\tau). \tag{10.11}$$

The kernel $K(\tau)$ in (10.11) is a function of bounded variation on $[0,\infty]$ and the integral is understood in the Stieltjes sense. It means in particular that both distributed and discrete delays can be used depending on the concrete choice of the kernel $K(\tau)$.

The initial condition for the system of (10.10) and (10.11) has the form

$$x(s)=\varphi(s), \qquad \dot{x}(s)=\dot{\varphi}(s), \quad s\le 0, \tag{10.12}$$

where $\varphi(s)$ is a given continuously differentiable function.

It is supposed also that system (10.10) is under the influence of stochastic perturbations of the type of white noise in the form

$$\ddot{x}(t)-\bigl(a+\sigma\dot{w}(t)\bigr)x(t)=u(t), \tag{10.13}$$

where $w(t)$ is the standard Wiener process, and σ is a constant.

Put $x_1(t)=x(t)$, $x_2(t)=\dot{x}(t)$. Then (10.11)–(10.13) can be represented in the form of the system

$$\begin{aligned} \dot{x}_1(t)&=x_2(t),\\ \dot{x}_2(t)&=ax_1(t)+\int_0^\infty \mathrm{d}K(\tau)\,x_1(t-\tau)+\sigma x_1(t)\dot{w}(t), \end{aligned} \tag{10.14}$$

with the initial condition $x_1(s)=\varphi(s)$, $x_2(s)=\dot{\varphi}(s)$, $s\le 0$.

Put

$$\begin{aligned} k_i&=\int_0^\infty \tau^i\,\mathrm{d}K(\tau), \quad i=0,1,\\ k_2&=\int_0^\infty \tau^2\bigl|\mathrm{d}K(\tau)\bigr|, \qquad a_1=-(a+k_0). \end{aligned} \tag{10.15}$$

The following theorem [247] gives a sufficient stability condition for the system (10.14).

Theorem 10.1 *Let*

$$a_1>0, \qquad k_1>0, \tag{10.16}$$

$$\sigma^2<2a_1\left(k_1-k_2\sqrt{\frac{a_1}{2(2-k_2)}}\right). \tag{10.17}$$

Then the trivial solution of the system (10.14) *is asymptotically mean square stable.*

Note [110] *that the inequalities* (10.16) *are the necessary conditions for asymptotic mean square stability of the trivial solution of the system* (10.14) *but the inequality* (10.17) *is only a sufficient one. Besides for condition* (10.17) k_2 *has to satisfy the inequality* $k_2<\sqrt{k^2+4k}-k<2$, *where* $k=k_1^2/a_1$.

Below, the mathematical model of the controlled inverted pendulum (10.10)–(10.12) *is considered in the following simple form*:

$$\ddot{x}(t)-\bigl(a+\sigma\dot{w}(t)\bigr)x(t)=b_1x(t-h_1)+b_2x(t-h_2), \quad t\ge 0. \tag{10.18}$$

Here $a>0$, b_1, b_2, $h_1>0$ *and* $h_2>0$ *are given arbitrary numbers. From* (10.15) *it follows that for* (10.18)

$$k_0=b_1+b_2, \qquad k_1=b_1h_1+b_2h_2,$$
$$k_2=|b_1|h_1^2+|b_2|h_2^2, \qquad a_1=-(a+b_1+b_2). \tag{10.19}$$

The main conclusion of our investigation here can be formulated in the following way: if conditions (10.16), (10.17) *and* (10.19) *hold, then the trivial solution of* (10.18) *is asymptotically mean square stable and there exists a small enough step of discretization of this equation that the trivial solution of the corresponding difference analogue is asymptotically mean square stable too. The estimation of the step of the discretization is also obtained.*

10.2.2 Construction of a Difference Analogue

Transform (10.18) into the system of the equations

$$\dot{x}(t)=y(t), \qquad \dot{y}(t)=ax(t)+\sum_{l=1}^{2}b_lx(t-h_l)+\sigma x(t)\dot{w}(t). \tag{10.20}$$

To construct a difference analogue of the system (10.20), put

$$\Delta>0, \qquad t_i=i\Delta, \qquad x_i=x(t_i), \qquad m_1=\frac{h_1}{\Delta}, \qquad m_2=\frac{h_2}{\Delta},$$
$$\xi_{i+1}=\frac{w(t_{i+1})-w(t_i)}{\sqrt{\Delta}}, \qquad \mathbf{E}\xi_i=0, \qquad \mathbf{E}\xi_i^2=1, \quad i=0,1,\dots, \tag{10.21}$$

Via the Euler–Maruyama scheme [184] a difference analogue of the system (10.20) is

$$x_{i+1}=x_i+\Delta y_i,$$
$$y_{i+1}=y_i+\Delta\left(ax_i+\sum_{l=1}^{2}b_lx_{i-m_l}\right)+\sigma\sqrt{\Delta}x_i\xi_{i+1}. \tag{10.22}$$

From the first equation of the system (10.22) we have

$$x_i=x_{i-m_l}+\Delta\sum_{j=1}^{m_l}y_{i-j}, \quad l=1,2. \tag{10.23}$$

From this and (10.19) it follows that

$$\sum_{l=1}^{2}b_lx_{i-m_l}=k_0x_i-\Delta\sum_{l=1}^{2}b_l\sum_{j=1}^{m_l}y_{i-j}. \tag{10.24}$$

Substituting (10.24) into the second equation of the system (10.22) and using (10.19) we obtain

$$y_{i+1} = y_i - \Delta a_1 x_i - \Delta^2 \sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} y_{i-j} + \sigma\sqrt{\Delta} x_i \xi_{i+1}. \tag{10.25}$$

Put

$$F_i = \Delta^2 \sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} (m_l + 1 - j) y_{i-j}. \tag{10.26}$$

Calculating $\hat{\Delta} F_i = F_{i+1} - F_i$ and using (10.26), (10.19) and (10.21) we have

$$\begin{aligned} \hat{\Delta} F_i &= \Delta^2 \sum_{l=1}^{2} b_l \left(\sum_{j=1}^{m_l} (m_l + 1 - j) y_{i+1-j} - \sum_{j=1}^{m_i} (m_l + 1 - j) y_{i-j} \right) \\ &= \Delta^2 \sum_{l=1}^{2} b_l \left(m_l y_i - \sum_{j=1}^{m_l} y_{i-j} \right) = \Delta k_1 y_i - \Delta^2 \sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} y_{i-j}. \end{aligned}$$

From this and (10.25) it follows that

$$y_{i+1} = -\Delta a_1 x_i + (1 - k_1 \Delta) y_i + \hat{\Delta} F_i + \sigma\sqrt{\Delta} x_i \xi_{i+1}.$$

So, the system (10.22) can be presented in the matrix form

$$z(i+1) = Az(i) + \hat{\Delta} F(i) + Bz(i)\xi_{i+1}, \tag{10.27}$$

where

$$\begin{aligned} z(i) &= \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad F(i) = \begin{pmatrix} 0 \\ F_i \end{pmatrix}, \\ A &= \begin{pmatrix} 1 & \Delta \\ -a_1 \Delta & 1 - k_1 \Delta \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \sigma\sqrt{\Delta} & 0 \end{pmatrix}. \end{aligned} \tag{10.28}$$

10.2.3 Stability Conditions for the Auxiliary Equation

Following the procedure of the construction of the Lyapunov functionals first consider the auxiliary equation without memory,

$$z(i+1) = Az(i) + Bz(i)\xi_{i+1} \tag{10.29}$$

and the function $v_i = z'(i) D z(i)$, where the matrix D is a positive definite solution of the matrix equation

$$A'DA - D = -C, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \ c > 0, \tag{10.30}$$

with the elements

$$d_{11} = \frac{a_1\Delta + k_1}{2a_1\Delta} + \frac{2 - k_1\Delta + a_1\Delta^2}{2}a_1 d_{22},$$
$$d_{12} = \frac{1}{2a_1\Delta} + \frac{a_1\Delta}{2}d_{22}, \quad d_{22} = \frac{2 - k_1\Delta + a_1\Delta^2 + 2a_1 c}{a_1\Delta(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}. \tag{10.31}$$

Calculating $\mathbf{E}\Delta v_i$ via (10.21) and (10.28)–(10.30), we have

$$\begin{aligned}
\mathbf{E}\Delta v_i &= \mathbf{E}\big[z'(i+1)Dz(i+1) - v_i\big] \\
&= \mathbf{E}\big[\big(Az(i) + Bz(i)\xi_{i+1}\big)'D\big(Az(i) + Bz(i)\xi_{i+1}\big) - v_i\big] \\
&= \mathbf{E}\big[-z'(i)Cz(i) + z'(i)B'DBz(i)\big] \\
&= -\big(1 - d_{22}\sigma^2\Delta\big)\mathbf{E}x_i^2 - c\mathbf{E}y_i^2.
\end{aligned} \tag{10.32}$$

So, if for some $c > 0$ the inequality

$$\sigma^2 < \frac{1}{d_{22}\Delta} \tag{10.33}$$

holds then the trivial solution of (10.29) is asymptotically mean square stable.

Note that (10.29) can also be written in the scalar form

$$x_{i+2} = A_0 x_{i+1} + A_1 x_i + \sigma_0 x_i \xi_{i+1} \tag{10.34}$$

with

$$A_0 = 2 - \Delta k_1, \qquad A_1 = \Delta k_1 - \Delta^2 a_1 - 1, \qquad \sigma_0 = \sigma\sqrt{\Delta^3}. \tag{10.35}$$

As follows from (5.7) the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of (10.34) are

$$|A_1| < 1, \quad |A_0| < 1 - A_1. \tag{10.36}$$

$$\sigma_0^2 < \frac{(1 + A_1)[(1 - A_1)^2 - A_0^2]}{1 - A_1}. \tag{10.37}$$

Substituting (10.35) into (10.36) we obtain the system of the inequalities

$$a_1\Delta < k_1, \qquad 4 - 2k_1\Delta + a_1\Delta^2 > 0,$$

with the solution

$$0 < \Delta < \begin{cases} a_1^{-1}k_1, & k_1^2 < 4a_1, \\ a_1^{-1}\big(k_1 - \sqrt{k_1^2 - 4a_1}\big), & k_1^2 \geq 4a_1. \end{cases} \tag{10.38}$$

Substituting (10.35) into (10.37) we obtain the condition

$$\sigma^2 < \frac{a_1(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}{2 - k_1\Delta + a_1\Delta^2}. \tag{10.39}$$

So, by the conditions (10.38) and (10.39) the trivial solution of (10.34) and (10.35) is asymptotically mean square stable.

From (10.31) and (10.33) it follows that if condition (10.39) holds then there exists a small enough $c > 0$ so that condition (10.33) holds too. Thus, the function $v_i = z'(i)Dz(i)$, where the matrix D is a positive definite solution of (10.30) is a Lyapunov function for (10.29).

10.2.4 Stability Conditions for the Difference Analogue

Let us obtain now a sufficient condition for asymptotic mean square stability of the trivial solution of (10.27). Rewrite this equation in the form

$$z(i+1) - F(i+1) = Az(i) - F(i) + Bz(i)\xi_{i+1}. \tag{10.40}$$

Following the procedure of the construction of the Lyapunov functionals we will construct a Lyapunov functional V_i for (10.40) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{1i} = \big(z(i) - F(i)\big)' D\big(z(i) - F(i)\big) \tag{10.41}$$

and the matrix D is a positive definite solution of the matrix equation (10.30) with elements (10.31).

Calculating $\mathbf{E}\Delta V_{1i}$ via (10.21), (10.30), (10.40) and (10.41) and similarly to (10.32) we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[\big(z(i+1) - F(i+1)\big)' D\big(z(i+1) - F(i+1)\big) - V_{1i}\big]\\ &= \mathbf{E}\big[\big(Az(i) - F(i) + Bz(i)\xi_{i+1}\big)' D\big(Az(i) - F(i) + Bz(i)\xi_{i+1}\big) - V_{1i}\big]\\ &= -\big(1 - d_{22}\sigma^2\Delta\big)\mathbf{E}x_i^2 - c\mathbf{E}y_i^2 - 2\mathbf{E}F'(i)D(A-I)z(i).\end{aligned} \tag{10.42}$$

Note that via (10.28)

$$\begin{aligned}&2F'(i)D(A-I)z(i)\\ &\quad = 2\Delta(0\ F_i)\begin{pmatrix} d_{11} & d_{12}\\ d_{12} & d_{22}\end{pmatrix}\begin{pmatrix} 0 & 1\\ -a_1 & -k_1\end{pmatrix}\begin{pmatrix} x_i\\ y_i\end{pmatrix}\\ &\quad = 2F_i\Delta\big[-a_1 d_{22}x_i + (d_{12} - k_1 d_{22})y_i\big]\end{aligned} \tag{10.43}$$

and we can put

$$\alpha = \frac{2 - k_1\Delta + a_1\Delta^2 + 2a_1 c}{(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)},$$
$$\beta = \frac{\Delta + c(2k_1 - a_1\Delta)}{(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}. \tag{10.44}$$

Then via (10.31) and (10.44)

$$d_{22} = \frac{\alpha}{a_1\Delta} \tag{10.45}$$

and

$$\begin{aligned}\Delta(d_{12} - k_1 d_{22}) &= \frac{1}{2a_1}\big(1 - \alpha(2k_1 - a_1\Delta)\big)\\ &= \frac{1}{2a_1}\left(1 - \frac{(2k_1 - a_1\Delta)(2 - k_1\Delta + a_1\Delta^2 + 2a_1 c)}{(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}\right) = -\beta.\end{aligned} \tag{10.46}$$

So, via (10.42)–(10.46)

$$\mathbf{E}\Delta V_{1i} = -\big(1 - a_1^{-1}\alpha\sigma^2\big)\mathbf{E}x_i^2 - c\mathbf{E}y_i^2 - 2\alpha\mathbf{E}x_i F_i - 2\beta\mathbf{E}y_i F_i.$$

Now put

$$q = \frac{1}{2}\sum_{l=1}^{2}|b_l|h_l(h_l + \Delta), \qquad S_i = \Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)y_{i-j}^2. \tag{10.47}$$

Using (10.26) and (10.21) and $\lambda > 0$ we have

$$2x_i F_i \le \Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)\left(\lambda x_i^2 + \frac{1}{\lambda}y_{i-j}^2\right) = \lambda q x_i^2 + \frac{S_i}{\lambda}$$

and analogously

$$2y_i F_i \le q y_i^2 + S_i.$$

As a result we obtain

$$\mathbf{E}\Delta V_{1i} \le -\big[1 - \alpha\big(\lambda q + a_1^{-1}\sigma^2\big)\big]\mathbf{E}x_i^2 - (c - \beta q)\mathbf{E}y_i^2 + \rho\mathbf{E}S_i,$$

where

$$\rho = \frac{\alpha}{\lambda} + \beta. \tag{10.48}$$

To neutralize the positive component in the estimation of ΔV_{1i} choose V_{2i} in the form

$$V_{2i} = \frac{1}{2}\rho\Delta^2 \sum_{l=1}^{2} |b_l| \sum_{j=1}^{m_l} (m_l + 1 - j)(m_l + 2 - j) y_{i-j}^2.$$

Then via (10.47)

$$\begin{aligned}\Delta V_{2i} &= \frac{1}{2}\rho\Delta^2 \sum_{l=1}^{2} |b_l| \sum_{j=1}^{m_l} (m_l + 1 - j)(m_l + 2 - j) y_{i+1-j}^2 - V_{2i} \\ &= \frac{1}{2}\rho\Delta^2 \sum_{l=1}^{2} |b_l| \sum_{j=0}^{m_l-1} (m_l - j)(m_l + 1 - j) y_{i-j}^2 - V_{2i} \\ &= \rho q y_i^2 - \rho S_i\end{aligned}$$

and for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\mathbf{E}\Delta V_i \le -\big[1 - \alpha\big(\lambda q + a_1^{-1}\sigma^2\big)\big]\mathbf{E}x_i^2 - \big[c - q(\beta + \rho)\big]\mathbf{E}y_i^2.$$

Using (10.48) we obtain the stability conditions in the form

$$\alpha\left(\lambda q + \frac{\sigma^2}{a_1}\right) < 1, \qquad q\left(2\beta + \frac{\alpha}{\lambda}\right) < c. \tag{10.49}$$

For $\lambda > 0$ from (10.49) it follows that

$$\frac{\alpha q}{c - 2\beta\, q} < \lambda < \frac{a_1 - \alpha\sigma^2}{\alpha q a_1}. \tag{10.50}$$

Thus, if

$$\frac{\alpha q}{c - 2\beta q} < \frac{a_1 - \alpha\sigma^2}{\alpha q a_1} \tag{10.51}$$

then there exists $\lambda > 0$ such that conditions (10.50) and therefore the conditions (10.49) hold.

Let us rewrite (10.51) in the form

$$\frac{\alpha^2 q^2}{c - 2\beta q} + \frac{\alpha\sigma^2}{a_1} < 1. \tag{10.52}$$

To stress the dependence of the left-hand part of (10.52) on c put

$$\begin{aligned}&A_0 = (k_1 - a_1\Delta)\big(4 - 2k_1\Delta + a_1\Delta^2\big), \\ &A_1 = 2 - k_1\Delta + a_1\Delta^2, \qquad A_2 = 2k_1 - a_1\Delta.\end{aligned} \tag{10.53}$$

Then via (10.44)

$$\alpha = \frac{A_1 + 2a_1 c}{A_0}, \qquad \beta = \frac{\Delta + A_2 c}{A_0}$$

and (10.52) takes the form

$$\frac{(A_1 + 2a_1 c)^2 q^2}{Qc - 2q\Delta} + 2\sigma^2 c < B, \tag{10.54}$$

where

$$B = A_0 - \frac{A_1}{a_1}\sigma^2, \qquad Q = A_0 - 2qA_2. \tag{10.55}$$

Transform (10.54) into the form

$$B_0 c^2 - B_1 c + B_2 < 0 \tag{10.56}$$

with

$$\begin{aligned} &B_0 = 4a_1^2 q^2 + 2Q\sigma^2, \\ &B_1 = BQ + 4q\sigma^2\Delta - 4A_1 a_1 q^2, \qquad B_2 = A_1^2 q^2 + 2qB\Delta. \end{aligned} \tag{10.57}$$

Minimization of the left-hand part of (10.56) with respect to c gives

$$B_1^2 > 4B_0 B_2. \tag{10.58}$$

Via (10.57) the inequality (10.58) can be represented as

$$\sigma^4 - 2P_1\sigma^2 + P_2 > 0, \tag{10.59}$$

where

$$P_1 = Ra_1, \quad P_2 = P_1^2 - \frac{8q^2 a_1^3 R}{Q}, \quad R = \frac{QA_0}{QA_1 + 4qa_1\Delta}. \tag{10.60}$$

From (10.59) and (10.60) by virtue of (10.54) and (10.55) it follows that

$$\sigma^2 < a_1\left(R - 2q\sqrt{\frac{2a_1 R}{Q}}\right). \tag{10.61}$$

Note that in condition (10.61) a_1 is defined in (10.19); q, Q and R are defined in (10.47), (10.55) and (10.60) and depend on Δ.

Thus, the following theorem is proven.

Theorem 10.2 *Let the parameters a, b_1, b_2, σ and Δ of the system (10.22) satisfy the conditions (10.16), (10.17) and (10.61). Then the trivial solution of the system (10.22) is asymptotically mean square stable.*

Lemma 10.1 *If for some values of the parameters* a, b_1, b_2 *and* σ *the conditions* (10.16) *and* (10.17) *hold*, *then there exists a small enough* $\Delta > 0$ *so that condition* (10.61) *holds too*.

Proof It is easy to see that for $\Delta = 0$ the condition (10.61) coincides with (10.17). Since the right-hand part of (10.61) is continuous with respect to Δ in the point $\Delta = 0$, if condition (10.61) holds for $\Delta = 0$ then it holds for some small enough $\Delta > 0$ too. The proof is completed. □

Corollary 10.1 *If the parameters* a, b_1, b_2 *and* σ *of the system* (10.22) *satisfy the conditions* (10.16) *and* (10.17) *then there exists a small enough* $\Delta > 0$ (*satisfying condition* (10.61)) *such that the trivial solution of the system* (10.22) *is asymptotically mean square stable*.

10.2.5 Nonlinear Model of the Controlled Inverted Pendulum

Consider the problem of stabilization for the nonlinear model of the controlled inverted pendulum

$$\ddot{x}(t) - \big(a + \sigma\dot{w}(t)\big)\sin x(t) = b_1 x(t-h_1) + b_2 x(t-h_2), \quad t \geq 0, \tag{10.62}$$

with the initial condition (10.12). Similarly to (10.22) the difference analogue of (10.62) is

$$\begin{aligned} x_{i+1} &= x_i + \Delta y_i, \\ y_{i+1} &= y_i + \Delta\left(ax_i + af(x_i) + \sum_{l=1}^{2} b_l x_{i-m_l}\right) + \sigma\sqrt{\Delta}\big(x_i + f(x_i)\big)\xi_{i+1}, \end{aligned} \tag{10.63}$$

where $f(x) = \sin x - x$. The system (10.22) is the linear part of system (10.63) and the order of nonlinearity of the system (10.63) equals 3, since $|f(x)| \leq \frac{1}{6}|x|^3$. Via [194] we obtain the following statement.

Corollary 10.2 *If the parameters* a, b_1, b_2, σ *and* Δ *of the system* (10.63) *satisfy conditions* (10.16), (10.17) *and* (10.61), *then the trivial solution of the system* (10.63) *is stable in probability*.

10.3 Difference Analogue of Nicholson's Blowflies Equation

The known Nicholson's blowflies differential equation (which is one of the most important mathematical models in ecology) with stochastic perturbations is considered. Stability of the positive equilibrium point of this nonlinear differential equation with delay and also the capability of its discrete analogue to preserve the stability properties of the original equation are studied. For this purpose, the equation considered is centered around the positive equilibrium point and linearized. Asymptotic mean square stability of the linear part of the considered equation is used to verify stability in probability of the nonlinear origin equation.

The necessary and sufficient condition for asymptotic mean square stability in the continuous case and a sufficient and the necessary and sufficient conditions for the discrete case are obtained from some known previous results.

The obtained stability regions are plotted in the plane of the parameters of the system. Stability conditions for the discrete analogue allow one to determinate an admissible step of discretization for numerical simulation of solution trajectories. The trajectories of stable and unstable solutions of the considered equation are simulated numerically in the deterministic and the stochastic cases for different values of the parameters and the initial data.

10.3.1 Nicholson's Blowflies Equation

Consider the nonlinear differential equation with delay and exponential nonlinearity

$$\dot{x}(t) = ax(t-h)\,\mathrm{e}^{-bx(t-h)} - cx(t). \tag{10.64}$$

It describes the population dynamics of Nicholson's blowflies. Here $x(t)$ is the size of the population at time t, a is the maximum per capita daily egg production rate, $1/b$ is the size at which the population reproduces at the maximum rate, c is the per capita daily adult death rate and h is the generation time.

Equation (10.64) is popular enough with researchers [30, 31, 57, 77, 95, 98, 114, 160, 167, 168, 172, 192, 254, 255, 269]. The majority of the results on (10.64) deal with the global attractiveness of the positive point of equilibrium and oscillatory behaviors of solutions [77, 97, 98, 114, 158–160, 167, 168, 172, 254]. In connection with numerical simulation of special interest is the investigation of discrete analogues of (10.64) [57, 114, 167, 255].

Below, we consider stability in probability of the positive equilibrium point of (10.1) by stochastic perturbations and also of one discrete analogue of this equation. The capability of a discrete analogue to preserve stability properties of the original differential equation is studied. It is shown that sufficient stability conditions obtained here for a discrete analogue are much better than the similar conditions obtained earlier in the deterministic case [114, 255]. All theoretical results are verified by numerical simulation. Besides it is shown that numerical simulation of the solution of difference analogue allows one to define more exactly a bound of stability region obtained by the sufficient stability condition.

The following method for investigation of the stability is used. The nonlinear equation considered is exposed to stochastic perturbations and is linearized in the neighborhood of the positive point of equilibrium. The conditions for asymptotic mean square stability of the trivial solution of the corresponding linear equation are obtained. Since the order of nonlinearity is higher than 1 these conditions are sufficient ones (both for continuous and discrete time [177, 194, 195, 227, 229]) for stability in probability of the initial nonlinear equation by stochastic perturbations. This method was used already for investigation of the stability of different nonlinear biological systems with delays: SIR epidemic model [24], predator–prey model [235] and other models [16, 41].

10.3.2 Stability Condition for the Positive Equilibrium Point

In (10.64) it is supposed that the parameters a, b and c are positive. By the conditions $c \geq a > 0$, $b > 0$, (10.64) has the trivial equilibrium point only, i.e. $x^* = 0$. By the conditions

$$a > c > 0, \qquad b > 0 \tag{10.65}$$

(10.64) has two points of equilibrium: the trivial one and a positive one

$$x^* = \frac{1}{b} \ln \frac{a}{c}. \tag{10.66}$$

It can be shown that the trivial equilibrium point in region (10.65) is unstable. The stability condition in region (10.65) for the positive equilibrium point (10.66) by stochastic perturbations is considered below.

As was proposed in [24, 235] and used later in [16, 30, 41, 199], let us assume that (10.64) is exposed to stochastic perturbations, which are of the type of white noise, are directly proportional to the distance of $x(t)$ from the point of equilibrium x^* and influence $\dot{x}(t)$ immediately. So, (10.64) is transformed into

$$\dot{x}(t) = ax(t-h)\,\mathrm{e}^{-bx(t-h)} - cx(t) + \sigma\big(x(t) - x^*\big)\dot{w}(t). \tag{10.67}$$

Let us center (10.67) on the positive point of equilibrium using the new variable $y(t) = x(t) - x^*$. In this way via (10.66) we obtain

$$\dot{y}(t) = -cy(t) + cy(t-h)\,\mathrm{e}^{-by(t-h)} + \frac{c}{b}\ln\frac{a}{c}\big(\mathrm{e}^{-by(t-h)} - 1\big) + \sigma y(t)\dot{w}(t). \tag{10.68}$$

It is clear that stability of the equilibrium point x^* of (10.67) is equivalent to stability of the trivial solution of (10.68).

Along with (10.68) we will consider the linear part of this equation. Using the representation $e^y = 1 + y + o(y)$ (where $o(y)$ means that $\lim_{y\to 0}\frac{o(y)}{y} = 0$) and neglecting by $o(y)$, we obtain the linear part (process $z(t)$) of (10.68) in the form

$$\dot{z}(t) = -cz(t) - c\left(\ln\frac{a}{c} - 1\right)z(t-h) + \sigma z(t)\dot{w}(t). \tag{10.69}$$

As is shown in [227, 229] if the order of nonlinearity of the equation under consideration is more than 1, then a sufficient condition for asymptotic mean square stability of the linear part of the initial nonlinear equation is also a sufficient condition for stability in probability of the initial equation. So, we will investigate sufficient conditions for asymptotic mean square stability of the linear part (10.69) of the nonlinear equation (10.68).

Via Lemma 1.5 the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (10.69) is

$$pG < 1, \tag{10.70}$$

where

$$p = \frac{\sigma^2}{2}, \qquad G = \begin{cases} \frac{1+\frac{c}{q}(\ln\frac{a}{c}-1)\sin(qh)}{c[1+(\ln\frac{a}{c}-1)\cos(qh)]}, & a > c\,\mathrm{e}^2,\ q = c\sqrt{\ln\frac{a}{c}(\ln\frac{a}{c}-2)}, \\ \frac{1+ch}{2c}, & a = c\,\mathrm{e}^2, \\ \frac{1+\frac{c}{q}(\ln\frac{a}{c}-1)\sinh(qh)}{c[1+(\ln\frac{a}{c}-1)\cosh(qh)]}, & c < a < c\,\mathrm{e}^2,\ q = c\sqrt{\ln\frac{a}{c}(2-\ln\frac{a}{c})}. \end{cases} \tag{10.71}$$

In particular, if $p > 0$, $h = 0$, then the stability condition (10.70) takes the form $c\ln\frac{a}{c} > p$; if $p = 0$, $h > 0$, then the region of stability is bounded by the lines $c = 0$, $c = a$ and $1 + (\ln\frac{a}{c} - 1)\cos(qh) = 0$ for $a > c\,\mathrm{e}^2$.

The conditions (10.70) and (10.71) give us regions (in the space of the parameters (a, c)) for asymptotic mean square stability of the trivial solution of (10.69) (and at the same time regions for stability in probability of the positive point of equilibrium x^* of (10.67)). In Fig. 10.1 the region of stability given by condition (10.70) and (10.71) is shown for $p = 0$, $h = 0$. Similar regions of stability are shown also for $p = 100$, $h = 0$ (Fig. 10.2) and for $p = 12$, $h = 0.024$ (Fig. 10.3).

Remark 10.2 Note that the stability conditions (10.70) and (10.71) have the following property: if the point (a, c) belongs to the stability region with some p and h, then for arbitrary positive α the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$ and $h_0 = \alpha^{-1}h$.

10.3.3 Stability of Difference Analogue

Consider a difference analogue of the nonlinear equation (10.68) using the Euler–Maruyama scheme [184]

$$y_{i+1} = (1 - c\Delta)y_i + c\Delta y_{i-k}\,\mathrm{e}^{-by_{i-k}} + \frac{c}{b}\ln\frac{a}{c}\Delta\big(\mathrm{e}^{-by_{i-k}} - 1\big) + \sigma\sqrt{\Delta}y_i\xi_{i+1}. \tag{10.72}$$

Here k is an integer, $\Delta = \frac{h}{k}$ is the step of discretization,

$$t_i = i\Delta, \quad y_i = y(t_i), \qquad \xi_{i+1} = \frac{1}{\sqrt{\Delta}}\big(w(t_{i+1}) - w(t_i)\big), \quad i = 0, 1, \ldots.$$

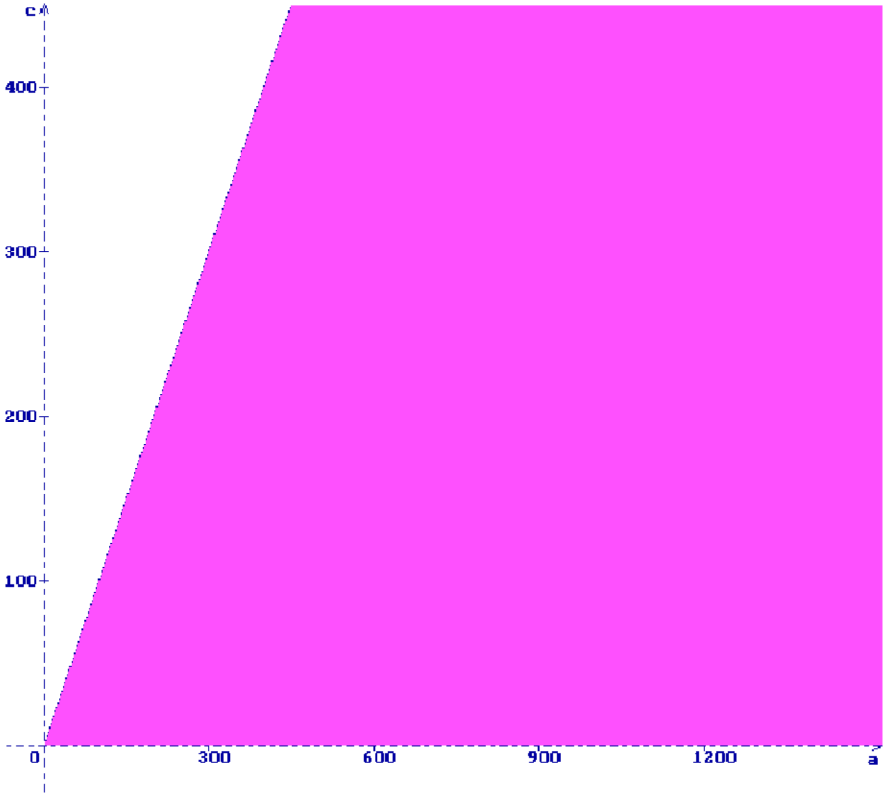


Fig. 10.1 Region of necessary and sufficient stability condition for (10.69): $p = 0$, $h = 0$

In compliance with (10.69) the linear part of (10.72) is

$$z_{i+1} = (1 - c\Delta)z_i + c\Delta\left(1 - \ln\frac{a}{c}\right)z_{i-k} + \sigma\sqrt{\Delta}z_i\xi_{i+1}. \tag{10.73}$$

Via Example 3.1 we obtain two sufficient conditions for asymptotic mean square stability of the trivial solution of (10.73):

$$\frac{p}{c} + \left|1 - \ln\frac{a}{c}\right||1 - c\Delta| + \frac{1}{2}c\Delta\left(1 + \left|1 - \ln\frac{a}{c}\right|^2\right) < 1 \tag{10.74}$$

and

$$\frac{p}{c} + \frac{1}{2}c\Delta\ln^2\frac{a}{c} < \left(1 - ch\left|1 - \ln\frac{a}{c}\right|\right)\ln\frac{a}{c}. \tag{10.75}$$

The regions for asymptotic mean square stability of the trivial solution of (10.73) (and at the same time regions for stability in probability of the trivial solution of (10.72)), obtained by the conditions (10.74) and (10.75), are shown in the space

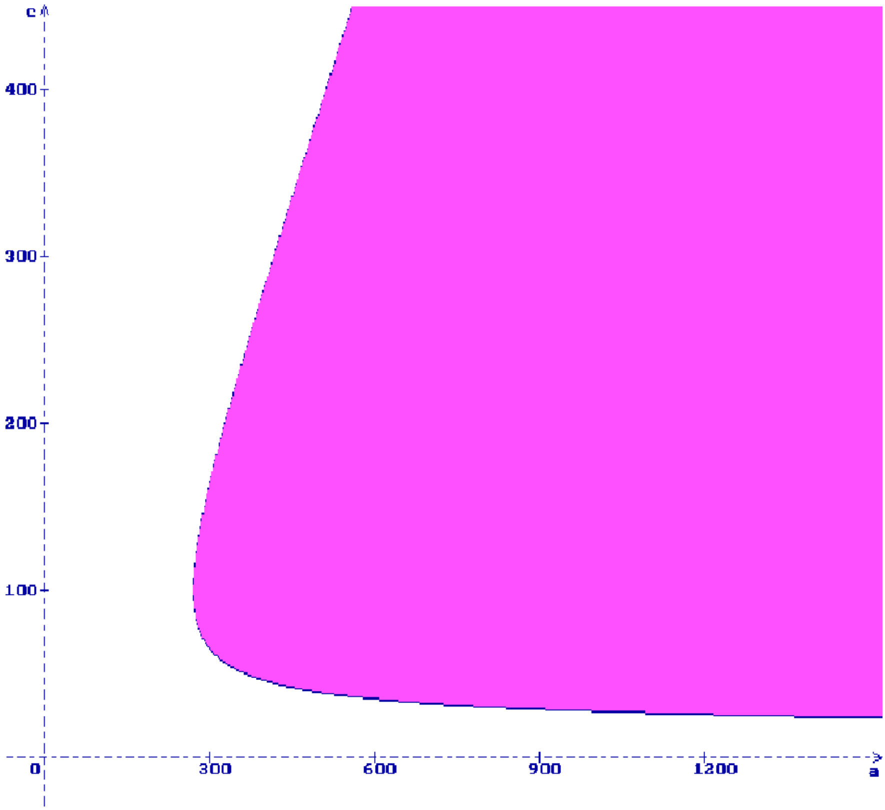


Fig. 10.2 Region of necessary and sufficient stability condition for (10.69): $p = 100$, $h = 0$

of the parameters (a, c) for $p = 12$, $h = 0.024$ and $\Delta = 0.004$ (Fig. 10.4), $\Delta = 0.006$ (Fig. 10.5), $\Delta = 0.008$ (Fig. 10.6), $\Delta = 0.012$ (Fig. 10.7). The main part (with number 1) of the stability region is obtained via the condition (10.74), the additional part (with number 2) is obtained via the condition (10.75).

Let us show in how far sufficient conditions (10.74) and (10.75) are close to the necessary and sufficient condition. Consider the case $p = 12, h = 0.024, \Delta = 0.012$. Since here $k = \frac{h}{\Delta} = 2$, we can use the necessary and sufficient stability condition (5.9). For (10.73) it can be represented in the form

$$\frac{p}{c} + \frac{1}{2}c\Delta\left[1 + \left(1 - \ln\frac{a}{c}\right)^2\right] + \frac{(1 - c\Delta)^2(1 - \ln\frac{a}{c})(1 - c\Delta\ln\frac{a}{c})}{1 - c\Delta(1 - \ln\frac{a}{c})(1 - c\Delta\ln\frac{a}{c})} < 1. \quad (10.76)$$

In Fig. 10.8 the stability region, obtained via the sufficient conditions (10.74) and (10.75) (number 1), is shown inside the stability region, obtained via the necessary and sufficient condition (10.76) (number 2).

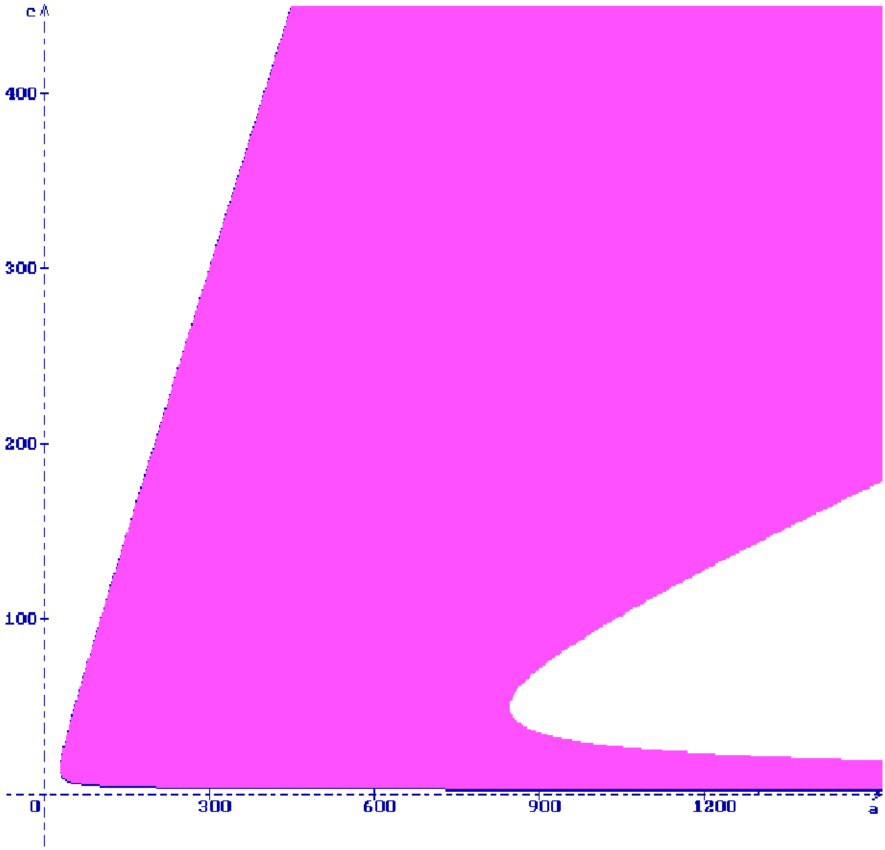


Fig. 10.3 Region of necessary and sufficient stability condition for (10.69): $p = 12$, $h = 0.024$

Remark 10.3 The conditions (10.74) and (10.75) can be represented in the explicit form, respectively:

$$c\,\mathrm{e}^{1+G_0} > a > c\,\mathrm{e}^{1-G_0}, \quad G_0 = \frac{\sqrt{1-\sigma^2\Delta} - |1 - c\Delta|}{c\Delta}, \tag{10.77}$$

and

$$c\,\mathrm{e}^{G_3} > a > \begin{cases} c\,\mathrm{e}^{G_1}, & c > c_0, \\ c\,\mathrm{e}^{G_2}, & c \le c_0, \end{cases}$$

$$c_0 = \frac{1 - \sqrt{1-\sigma^2\Delta}}{2\Delta}, \qquad G_1 = \frac{ch - 1 + \sqrt{(1-ch)^2 + (2h-\Delta)\sigma^2}}{c(2h-\Delta)}, \tag{10.78}$$

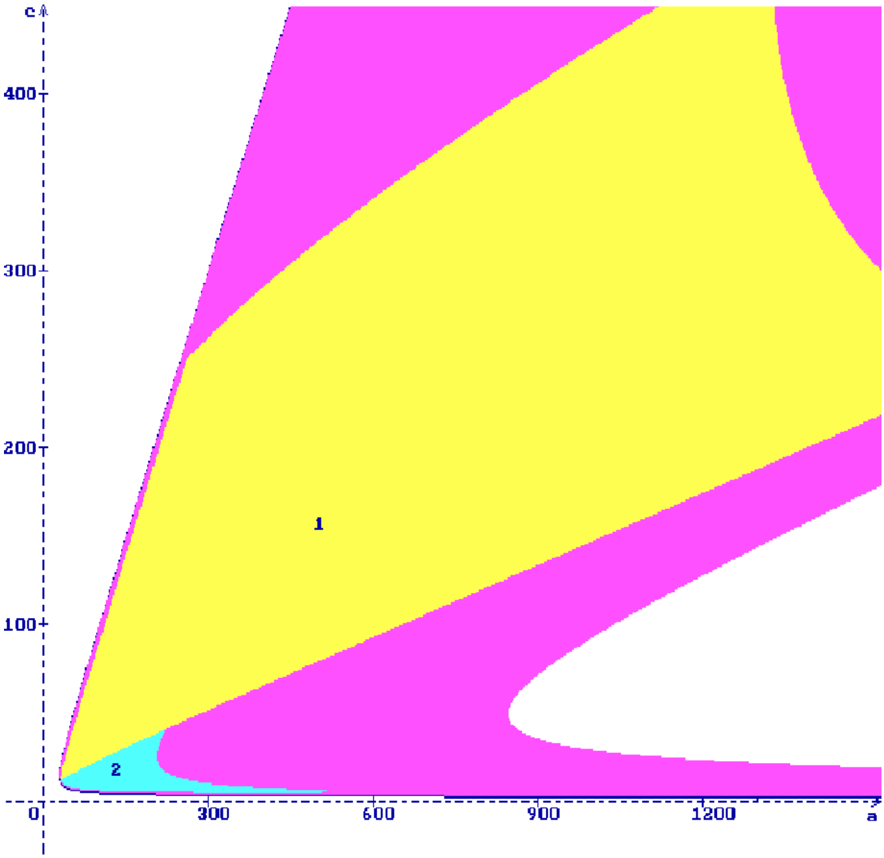


Fig. 10.4 Region of sufficient stability condition for (10.73): $p = 12$, $h = 0.024$, $\Delta = 0.004$

$$G_2 = \frac{ch + 1 - \sqrt{(1 + ch)^2 - (2h + \Delta)\sigma^2}}{c(2h + \Delta)},$$

$$G_3 = \frac{ch + 1 + \sqrt{(1 + ch)^2 - (2h + \Delta)\sigma^2}}{c(2h + \Delta)}.$$

Remark 10.4 Conditions (10.74), (10.75) and (10.76) for arbitrary values of the parameters of (10.67) allow us to choose the admissible step of discretization Δ by numerical simulation of the stable solution of this equation. For example, in Figs. 10.4 and 10.5 we can see that for the simulation of (10.67) and the solution with $a = 900$, $c = 200$ we can use $\Delta = 0.004$ or $\Delta = 0.006$. But taking into account Figs. 10.6 and 10.7 we cannot be sure that it is possible to use $\Delta = 0.008$ or $\Delta = 0.012$.

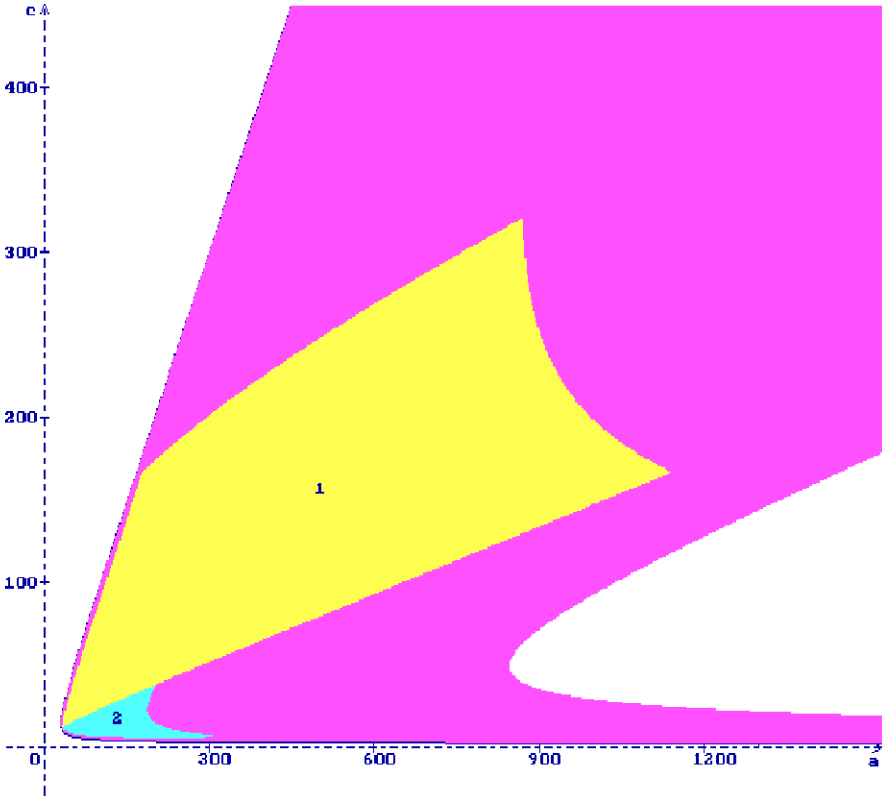


Fig. 10.5 Region of sufficient stability condition for (10.73): $p = 12$, $h = 0.024$, $\Delta = 0.006$

Remark 10.5 Note that the stability conditions (10.74) and (10.75) have the following property: if the point (a, c) belongs to the stability region with some p, h and Δ, then for arbitrary positive α the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$, $h_0 = \alpha^{-1} h$ and $\Delta_0 = \alpha^{-1} \Delta$.

Remark 10.6 In [114, 255] the discrete analogue of (10.64) was considered in the form (in our notation)

$$x_{i+1} = (1 - c\Delta)x_i + a\Delta x_{i-k}\, \mathrm{e}^{-bx_{i-k}}.$$

By the assumption $c\Delta < 1$ the sufficient condition for asymptotic stability of positive equilibrium (10.66) was obtained in [114]:

$$\left[(1 - c\Delta)^{-(k+1)} - 1\right]\left(\frac{a}{c} - 1\right) < 1 \tag{10.79}$$

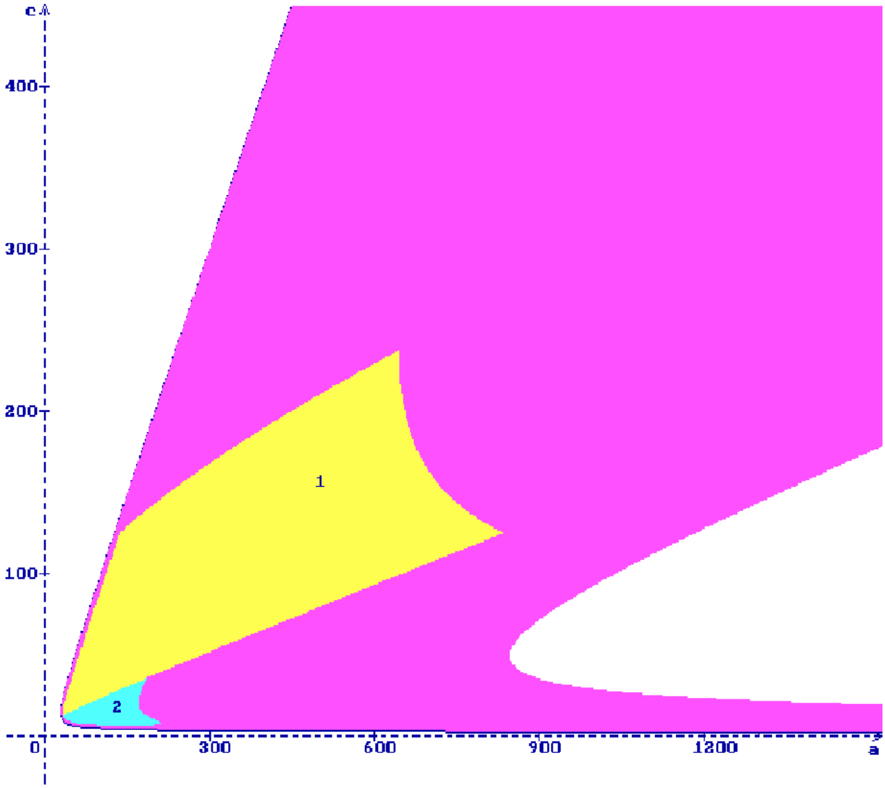


Fig. 10.6 Region of sufficient stability condition for (10.73): $p = 12$, $h = 0.024$, $\Delta = 0.008$

and improved in [255]:

$$\left[(1 - c\Delta)^{-(k+1)} - 1\right] \ln \frac{a}{c} \le 1. \tag{10.80}$$

Note that in the conditions (10.77) and (10.78) the assumption $c\Delta < 1$ does not have to be made. Let us show that even with the assumption $c\Delta < 1$ the conditions (10.77) and (10.78) (in deterministic case, i.e. by $\sigma^2 = 0$) are better than (10.80).

In fact, if $\sigma^2 = 0$ and $c\Delta < 1$, then the condition (10.77) takes the form $a < c\,e^2$. Representing (10.80) as

$$a \le c\,e^{((1-c\Delta)^{-(k+1)}-1)^{-1}} \tag{10.81}$$

one can see that (10.77) is better than (10.81) if $((1 - c\Delta)^{-(k+1)} - 1)^{-1} \le 2$ or $c\Delta \ge 1 - (\frac{2}{3})^{\frac{1}{k+1}}$.

Let us show that condition (10.78) is better than (10.80) for $c\Delta \in (0, 1)$. In fact, if $\sigma^2 = 0$, then condition (10.78) takes the form

$$a < c\,e^{\frac{ch+1}{c(h+0.5\Delta)}}. \tag{10.82}$$

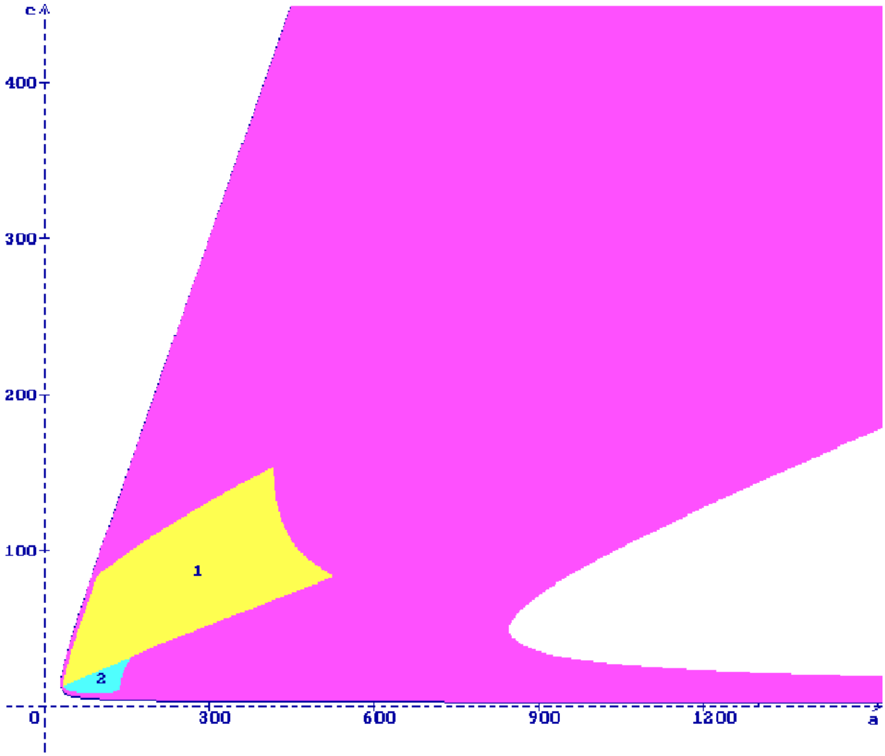


Fig. 10.7 Region of sufficient stability condition for (10.73); $p = 12$, $h = 0.024$, $\Delta = 0.012$

Via (10.81) and (10.82) it is enough to show that

$$\frac{1}{(1 - c\Delta)^{-(k+1)} - 1} \leq \frac{ch + 1}{c(h + 0.5\Delta)}$$

or the function

$$f(c) = \frac{1}{(1 - c\Delta)^{k+1}} - 1 - \frac{c(h + 0.5\Delta)}{ch + 1}$$

is nonnegative for $c\Delta \in [0, 1)$. This is in fact so, since $f(0) = 0$ and via $k\Delta = h$

$$f'(c) = \frac{h + \Delta}{(1 - c\Delta)^{k+2}} - \frac{h + 0.5\Delta}{(ch + 1)^2} \geq 0.$$

In Fig. 10.9 one can see the stability regions for $h = 0.024$ and $\Delta = 0.012$ given by the condition (10.79) (number 1), given by the condition (10.80) (numbers 1 and 2), given by the conditions (10.74), (10.75) (numbers 1, 2 and 3) and given by the condition (10.76) (numbers 1, 2, 3 and 4).

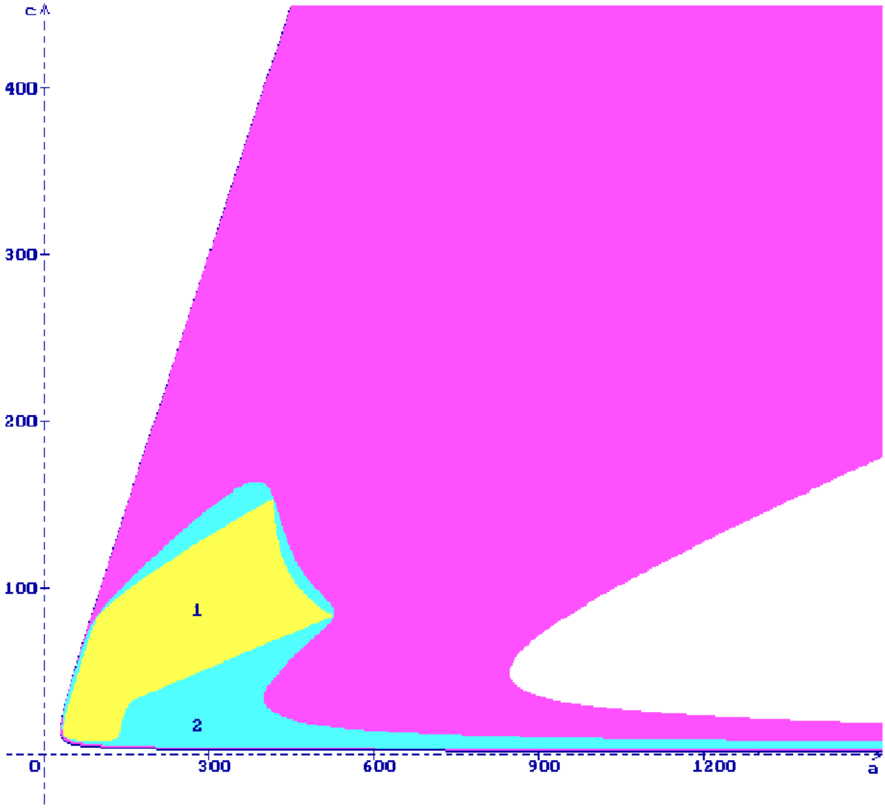


Fig. 10.8 Regions of sufficient stability condition and necessary and sufficient stability condition for (10.73): $p = 12$, $h = 0.024$, $\Delta = 0.012$

10.3.4 Numerical Analysis in the Deterministic Case

Consider (10.67) at first in the deterministic case ($p = 0$) with delay $h = 0.024$. We will simulate solutions of this equation via its discrete analogue (10.73) with $\Delta = 0.012$. The corresponding stability region is shown in Fig. 10.10. Note that for $p = 0$ the stability region slightly differs from the similar region for $p = 12$ (Fig. 10.8). The initial function is $z(s) = a_0 \cos(s)$, $s \in [-h, 0]$, where a_0 has different values in different points.

In Fig. 10.10 one can see the points $A(520, 100)$, $B(529.45, 100)$, $C(540, 100)$, $D(544.5, 46)$, $E(544.5, 40)$, $F(544.5, 34)$, $K(279.9, 150)$, $L(87.5, 85)$, $M(40, 40)$. The trajectories of solutions of (10.73) in these points are shown, respectively, in Fig. 10.11 (A, $a_0 = 5$), Fig. 10.12 (B, $a_0 = 5$), Fig. 10.13 (C, $a_0 = 0.1$), Fig. 10.14 (D, $a_0 = 0.4$), Fig. 10.15 (E, $a_0 = 4$), Fig. 10.16 (F, $a_0 = 5$), Fig. 10.17 (K, $a_0 = 6$), Fig. 10.18 (L, $a_0 = 5$) and Fig. 10.19 (M, $a_0 = 3$).

The points A and F belong to the stability region; the solutions of (10.73) in these points are stable. The points C and D do not belong to the stability region;

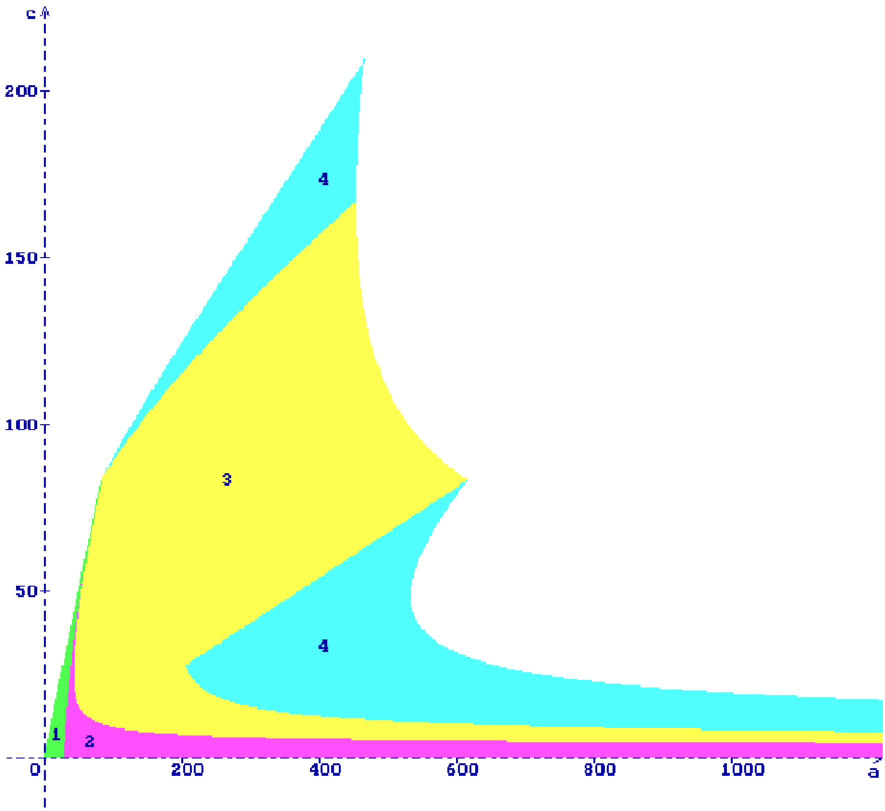


Fig. 10.9 Regions of sufficient stability condition and necessary and sufficient stability condition for (10.73): $p = 0$, $h = 0.024$, $\Delta = 0.012$

the solutions of (10.73) in these points are unstable. The points B, E, K, L and M are placed on the bound of the stability region; the solutions of (10.73) in these points do not converge to zero but converge to bounded functions, in particular to a constant as in the point M. Note, however, that in the point M (i.e., in the case $b > 0$, $a = c > 0$) the initial equation (10.64) has only a zero equilibrium.

Comparing the solutions of (10.73) in the points A, B, C and in the points D, E, F, one can see also that to move a bit outside of the stability region gives an unstable solution and to move a bit inside of the stability region gives a stable solution. A similar result can be obtained comparing the solution of (10.73) in the point $L(87.5, 85)$ (Fig. 10.18) with the solutions in the points $L_1(88, 85)$ (Fig. 10.20) and $L_2(87, 85)$ (Fig. 10.21).

This fact can be used to construct the exact bound of the stability region in the case when we have a sufficient stability condition only. For example, in the case $p = 0$, $h = 0.024$, $\Delta = 0.008$ the points $P(50, 50)$, $Q(288.65, 170)$, $R(680, 250.079)$, $S(810, 170)$, $T(923.63, 125)$, $U(652.6, 50)$, $V(1000, 24.16)$ (Fig. 10.22) belong to the bound of the stability region, since in all these points the

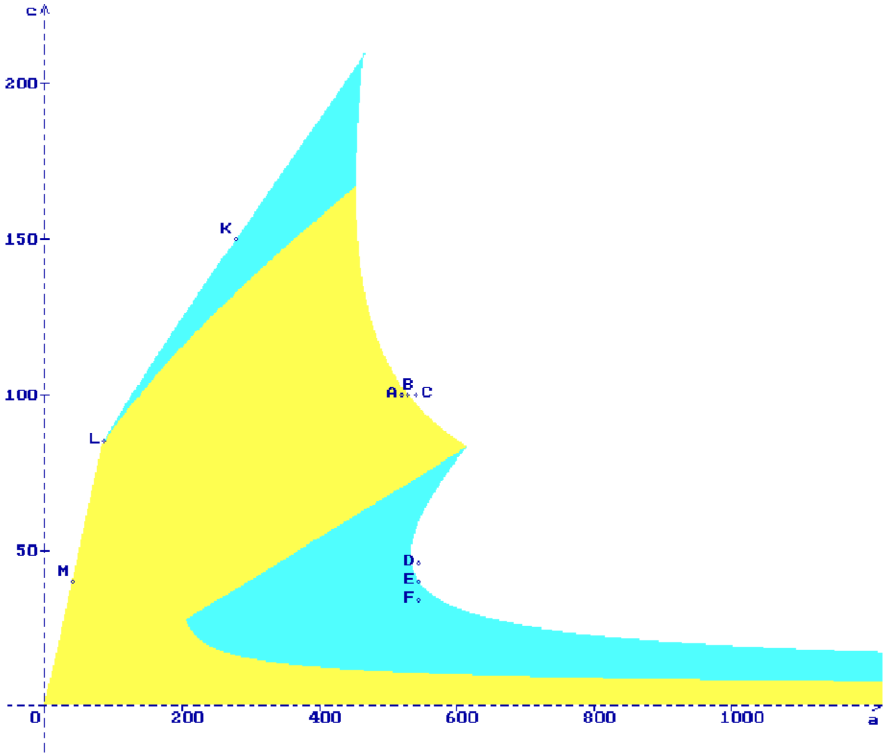


Fig. 10.10 Regions of sufficient stability condition and necessary and sufficient stability condition for (10.73): $p = 0$, $h = 0.024$, $\Delta = 0.012$

solutions of (10.73) do not converge to zero but are bounded functions (Fig. 10.23 (P, $a_0 = 3$), Fig. 10.24 (Q, $a_0 = 5$), Fig. 10.25 (R, $a_0 = 5$), Fig. 10.26 (S, $a_0 = 5$), Fig. 10.27 (T, $a_0 = 5$), Fig. 10.28 (U, $a_0 = 4$), Fig. 10.29 (V, $a_0 = 4$)). Note that in the point P similarly to the point M the initial equation (10.64) has only a zero equilibrium.

To illustrate Remark 10.4 consider the point $A_0(5.2, 1)$ which corresponds to the point $A(520, 100)$ with $\alpha = 0.01$. The solution of (10.73) in the point A_0 is stable with $h = 2.4$, $\Delta = 1.2$ (Fig. 10.30, $a_0 = 5$). Note that in spite of the fact that (10.73) has the same coefficients in both these cases the graphic of the solution in the point A_0 differs from the graphic of the solution in the point A (Fig. 10.11, $a_0 = 5$), since the initial functions in both cases are different. In the point A it is $z_{-2} = 5\cos(0.024)$, $z_{-1} = 5\cos(0.012)$, $z_0 = 5$, in the point A_0 it is $z_{-2} = 5\cos(10.13)$, $z_{-1} = 5\cos(10.2)$, $z_0 = 5$.

Consider now the behavior of the solution of nonlinear equation (10.68) in the case $p = 0$. We will simulate solutions of this equation via its discrete analogue (10.72) with $\Delta = 0.012$. If in the point (a, c) the trivial solution of (10.73) is asymptotically stable (it means that for arbitrary initial function the solution of (10.73) goes to zero) then the trivial solution of (10.72) is stable in the first approximation

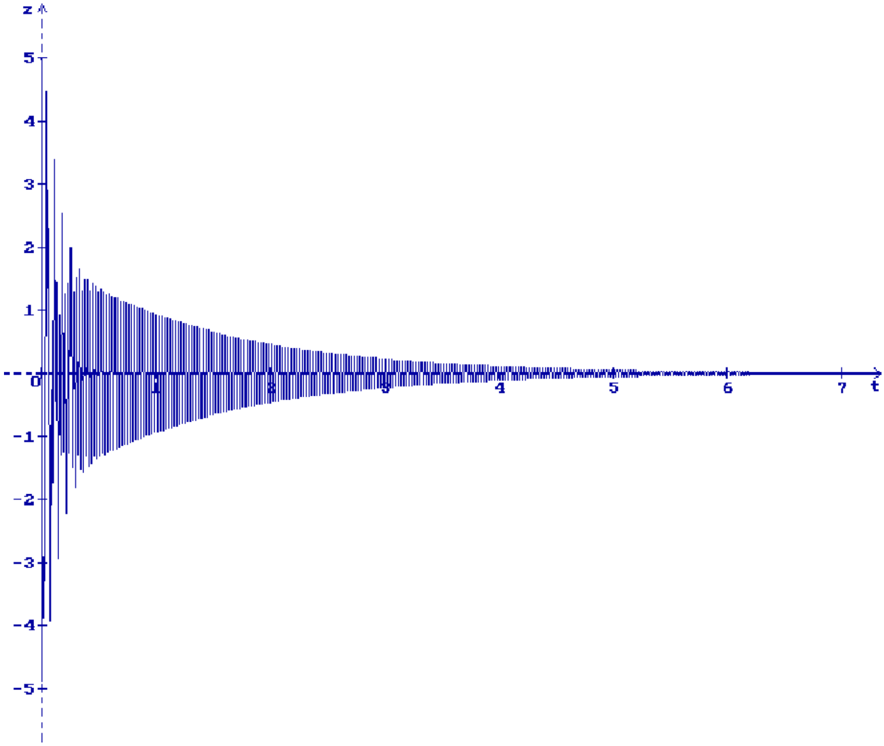


Fig. 10.11 Stable solution of (10.73) in the point $A(520, 100)$, $a_0 = 5$

(it means that for each small enough initial function the solution of (10.72) goes to zero). On the other hand if the trivial solution of (10.73) is not asymptotically stable, then for arbitrary indefinitely small initial function the solution of (10.72) does not go to zero.

These facts are illustrated by the following examples. In the point $A(520, 100)$ the trivial solution of (10.73) is asymptotically stable (Fig. 10.11, $a_0 = 5$), so in this point the solution of (10.72) ($b = 4$) goes to zero for small enough initial function (Fig. 10.31, $a_0 = 0.437$) and quickly enough goes to infinity for a little larger initial function (Fig. 10.32, $a_0 = 0.438$). In the point $C(540, 100)$ the trivial solution of (10.73) is not asymptotically stable (Fig. 10.13, $a_0 = 0.1$) and the solution of (10.72) ($b = 1$) goes to infinity for an indefinitely small initial function (Fig. 10.33, $a_0 = 0.001$).

10.3.5 Numerical Analysis in the Stochastic Case

Consider finally the behavior of the solution of (10.69) in the stochastic case with $p = 12$, delay $h = 0.024$ and the initial function $z(s) = a_0 \cos(s)$, $s \in [-h, 0]$.

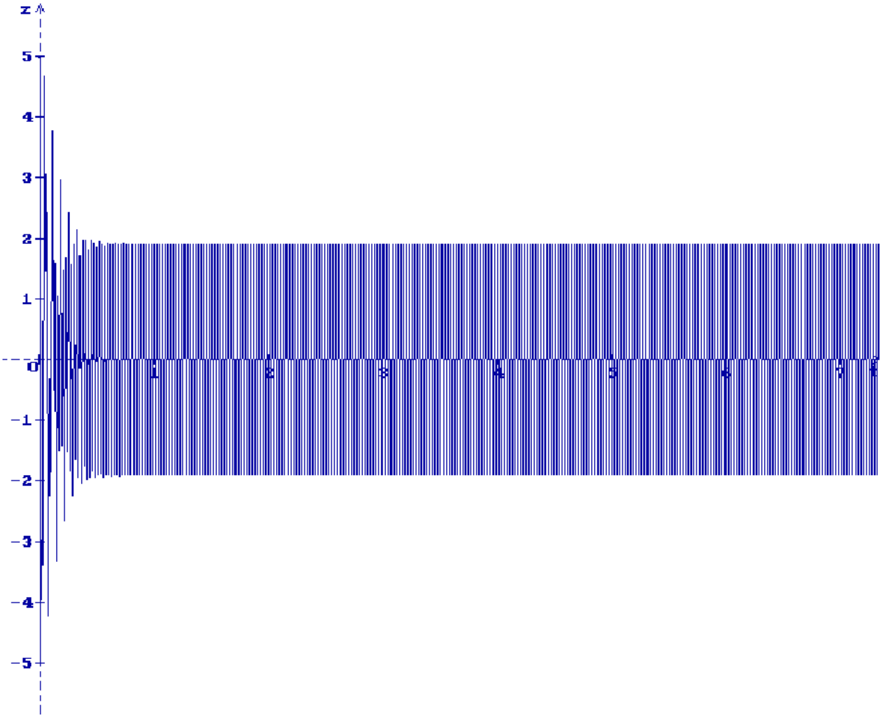


Fig. 10.12 Bounded solution of (10.73) in the point $B(529.45, 100)$, $a_0 = 5$

A solution of this equation is simulated here via its discrete analogue (10.73) with $\Delta = 0.012$. The corresponding stability region is shown in Fig. 10.34, which is the increasing copy of Fig. 10.8 with the additional points $X(160, 100)$, $Y(465, 100)$, which belong to the stability region of (10.73), and the points $W(120, 100)$, $Z(510, 100)$, which do not belong to the stability region of (10.73).

For numerical simulation of the solution of (10.73) one uses some special algorithm of numerical simulation of the Wiener process trajectories [209]. Fifty trajectories of the Wiener process obtained via this algorithm are shown in Fig. 10.35. In Fig. 10.36 ten trajectories of the solution of (10.73) are shown in the point W with $a_0 = 0.1$. The point W belongs to the stability region of stochastic differential equation (10.69), but it does not belong to the stability region of its difference analogue (10.73). One can see that each trajectory of the solution of (10.73) in the point W oscillates and goes to infinity.

A similar picture can be seen in Fig. 10.37 where 100 trajectories of the solution of (10.73) are shown in the point Z with $a_0 = 0.1$. In Fig. 10.38 100 trajectories of the solution of (10.73) are shown in the point X with $a_0 = 8.5$. The point X belongs to the stability region of (10.73) and all trajectories go to zero. One hundred trajectories of the stable solution of (10.73) are shown also in Fig. 10.39 in the point Y with $a_0 = 6.5$.

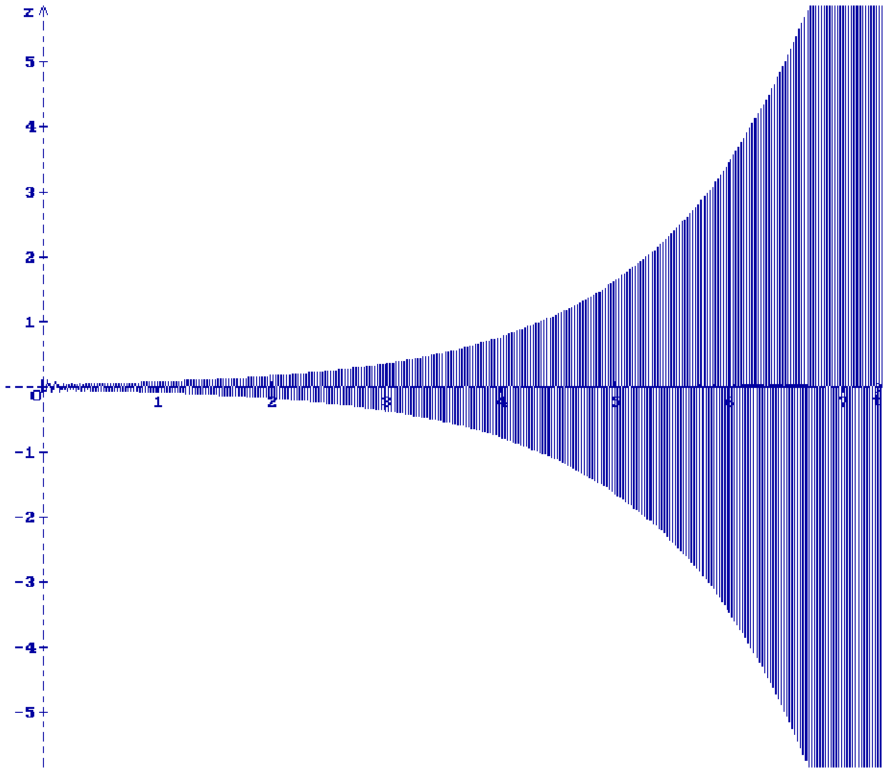


Fig. 10.13 Unstable solution of (10.73) in the point $C(540, 100)$, $a_0 = 0.1$

10.4 Difference Analogue of the Predator–Prey Model

Consider now the problem, which is similar to the problem from the previous section, but for a more complicated (two-dimensional) mathematical model type of the predator–prey model:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\big(a - a_1x_1(t) - a_2x_2(t)\big), \\ \dot{x}_2(t) &= -bx_2(t) + b_1x_1(t-h_1)x_2(t-h_2), \end{aligned} \tag{10.83}$$

$$x_m(s) = \varphi_m(s), \quad s \in \big[-\max(h_1, h_2), 0\big],\ m = 1, 2. \tag{10.84}$$

Here $x_1(t)$ and $x_2(t)$ are the densities of prey and predator populations, respectively, and h_1 and h_2 are nonnegative delays.

Note that mathematical models of the type of (10.83) or more general ones, with both continuous time and discrete time are interesting enough for a lot of researchers ([16, 20, 36, 42, 49, 56, 63, 74–76, 83, 84, 93, 101, 102, 105, 171, 173, 175, 183, 190, 202, 235, 246, 265–268, 272, 275, 277]).

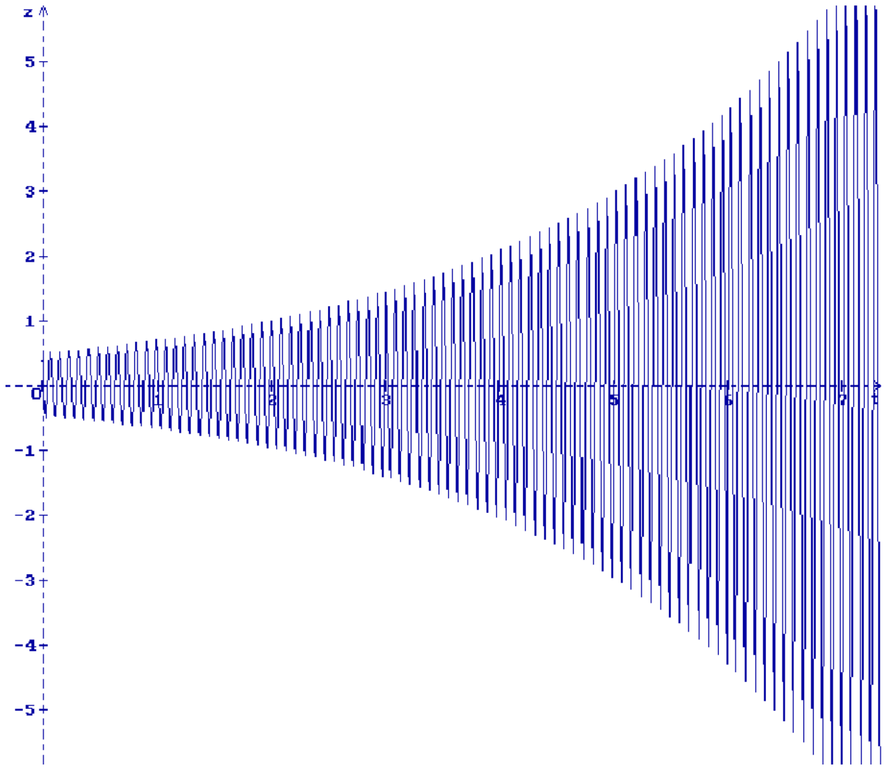


Fig. 10.14 Unstable solution of (10.73) in the point $D(544.5, 46)$, $a_0 = 0.4$

10.4.1 Positive Equilibrium point, Stochastic Perturbations, Centering and Linearization

Suppose that the parameters a, a_1, a_2, b, b_1 in the system (10.83) are positive constants such that

$$A = a - b\frac{a_1}{b_1} > 0. \tag{10.85}$$

Then the system (10.83) has a positive point of equilibrium (x_1^*, x_2^*), which is defined by the conditions $\dot{x}_1(t) \equiv 0$, $\dot{x}_2(t) \equiv 0$, and which has the form

$$x_1^* = \frac{b}{b_1}, \qquad x_2^* = \frac{A}{a_2}. \tag{10.86}$$

As was assumed above, the system (10.83) is exposed to stochastic perturbations that are directly proportional to the deviation of the state of the system $(x_1(t), x_2(t))$ from the point of equilibrium (x_1^*, x_2^*) and have influence on $\dot{x}_1(t)$, $\dot{x}_2(t)$ such that

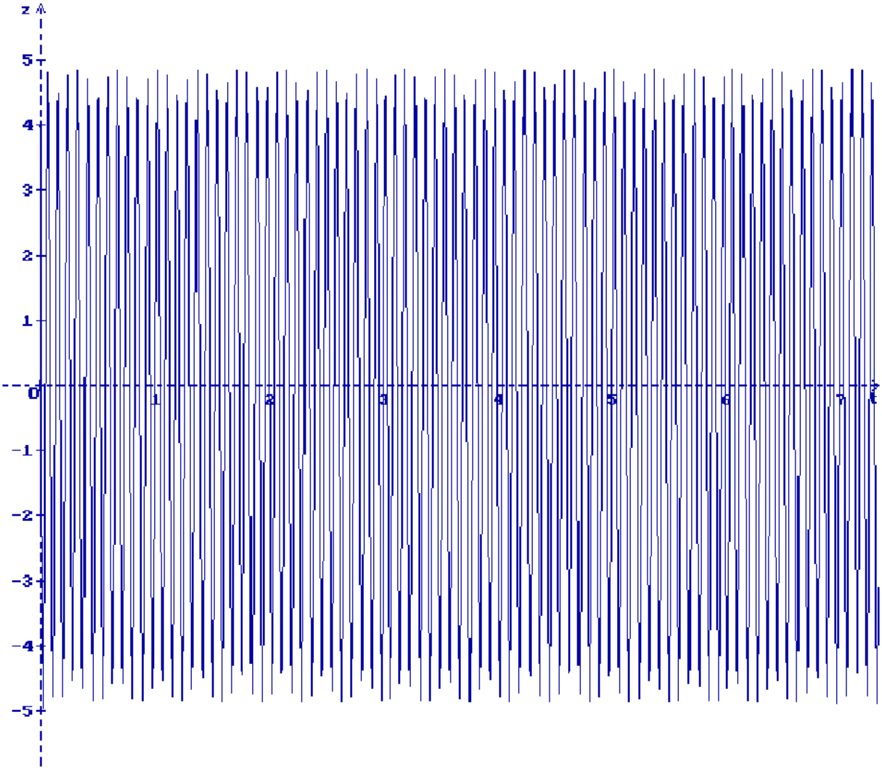


Fig. 10.15 Bounded solution of (10.73) in the point $E(544.5, 40)$, $a_0 = 4$

the system (10.83) takes the form

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big(a - a_1 x_1(t) - a_2 x_2(t)\big) + \sigma_1\big(x_1(t) - x_1^*\big)\dot{w}_1(t),\\ \dot{x}_2(t) &= -bx_2(t) + b_1 x_1(t-h_1)x_2(t-h_2) + \sigma_2\big(x_2(t) - x_2^*\big)\dot{w}_2(t).\end{aligned} \tag{10.87}$$

Here σ_1 and σ_2 are arbitrary constants, $w_1(t)$ and $w_2(t)$ are independent standard Wiener processes. Note that the equilibrium point (x_1^*, x_2^*) of the system (10.83) is also a solution of the system (10.87).

To center the system (10.87) around the equilibrium point (x_1^*, x_2^*) consider the new variables $y_1(t) = x_1(t) - x_1^*$, $y_2(t) = x_2(t) - x_2^*$. The system (10.87) takes the form

$$\begin{aligned}\dot{y}_1(t) &= -\big(y_1(t) + x_1^*\big)\big(a_1 y_1(t) + a_2 y_2(t)\big) + \sigma_1 y_1(t)\dot{w}_1(t),\\ \dot{y}_2(t) &= -by_2(t) + b_1\big(x_2^* y_1(t-h_1) + x_1^* y_2(t-h_2)\\ &\quad + y_1(t-h_1)y_2(t-h_2)\big) + \sigma_2 y_2(t)\dot{w}_2(t),\\ y_m(s) &= \varphi_m(s), \quad s \in \big[-\max(h_1, h_2), 0\big],\ m = 1, 2.\end{aligned} \tag{10.88}$$

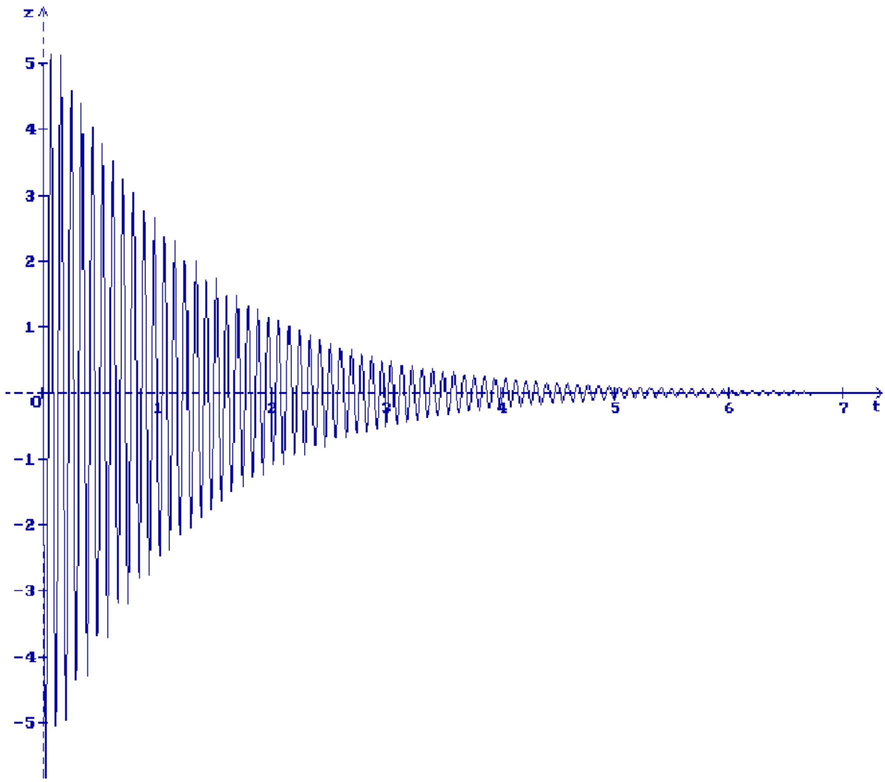


Fig. 10.16 Stable solution of (10.73) in the point $F(544.5, 34)$, $a_0 = 5$

It is evident that stability of the equilibrium point (x_1^*, x_2^*) of the system (10.87) is equivalent to stability of the trivial solution of the system (10.88).

Rejecting the nonlinear elements in the system (10.88) we obtain the system

$$\begin{aligned}
\dot{z}_1(t) &= -x_1^*\big(a_1 z_1(t) + a_2 z_2(t)\big) + \sigma_1 z_1(t)\dot{w}_1(t),\\
\dot{z}_2(t) &= -b z_2(t) + b_1\big(x_2^* z_1(t-h_1) + x_1^* z_2(t-h_2)\big) + \sigma_2 z_2(t)\dot{w}_2(t),\\
z_m(s) &= \varphi_m(s), \quad s \in \big[-\max(h_1,h_2), 0\big],\ m = 1,2,
\end{aligned} \tag{10.89}$$

which we will consider as a linear part of the system (10.88).

Put $y(t) = (y_1(t), y_2(t))$, $z(t) = (z_1(t), z_2(t))$, $\varphi(s) = (\varphi_1(s), \varphi_2(s))$.

Via the general method of the construction of the Lyapunov functionals for stochastic differential equations [126, 127, 129, 133, 134, 136, 141, 182, 227, 229–231] sufficient conditions for asymptotic mean square stability of the trivial solution of the system (10.89) were obtained in [246].

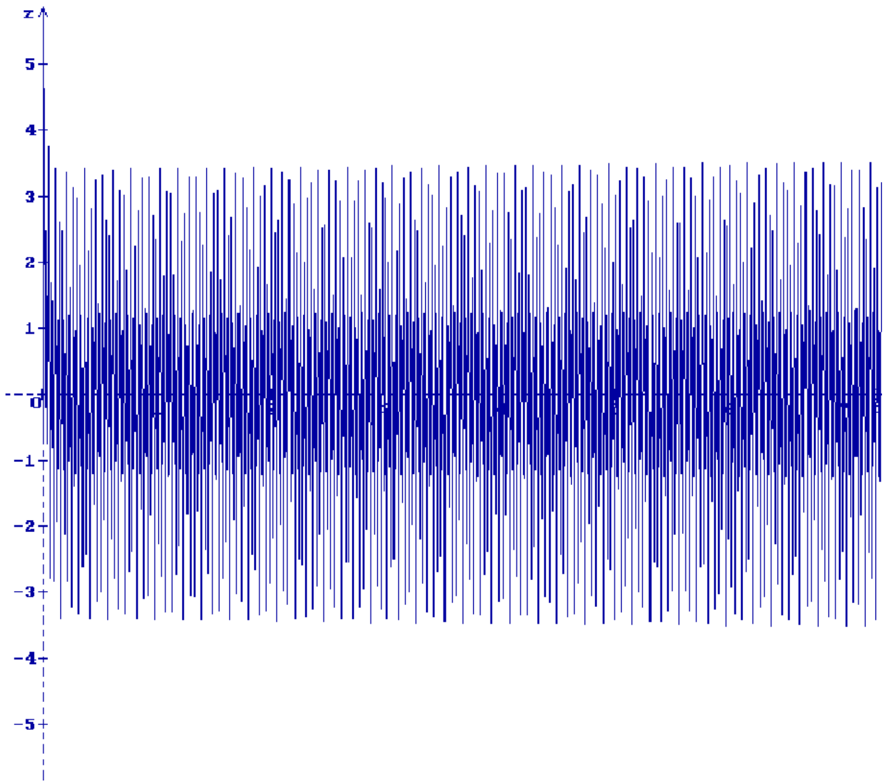


Fig. 10.17 Bounded solution of (10.73) in the point $K(279.9, 150)$, $a_0 = 6$

Theorem 10.3 *Put*

$$A_1 = \frac{a_2^2 b\sigma_2^2}{Ab_1^2}, \qquad A_2 = 2b\left(\frac{a_1}{b_1} - Ah_1\right) - \sigma_1^2, \qquad A_3 = \frac{2Ab_1}{a_2},$$
$$A_4 = \frac{a_2 b}{b_1}\left(2(1 - bh_2) - \frac{a_1\sigma_2^2}{Ab_1}\right), \qquad B_1 = \frac{a_2 b^2}{b_1}h_2, \qquad B_2 = Abh_1. \tag{10.90}$$

If the conditions (10.85),

$$h_1 < \frac{1}{A}\left(\frac{a_1}{b_1} - \frac{\sigma_1^2}{2b}\right), \qquad h_2 < \frac{1}{b}\left(1 - \frac{a_1\sigma_2^2}{2Ab_1}\right),$$
$$\left(\sqrt{B_1^2 + A_1A_2} + B_1\right)\left(\sqrt{B_2^2 + A_3A_4} + B_2\right) < A_2A_4, \tag{10.91}$$

hold, then the trivial solution of the system (10.89) *is asymptotically mean square stable.*

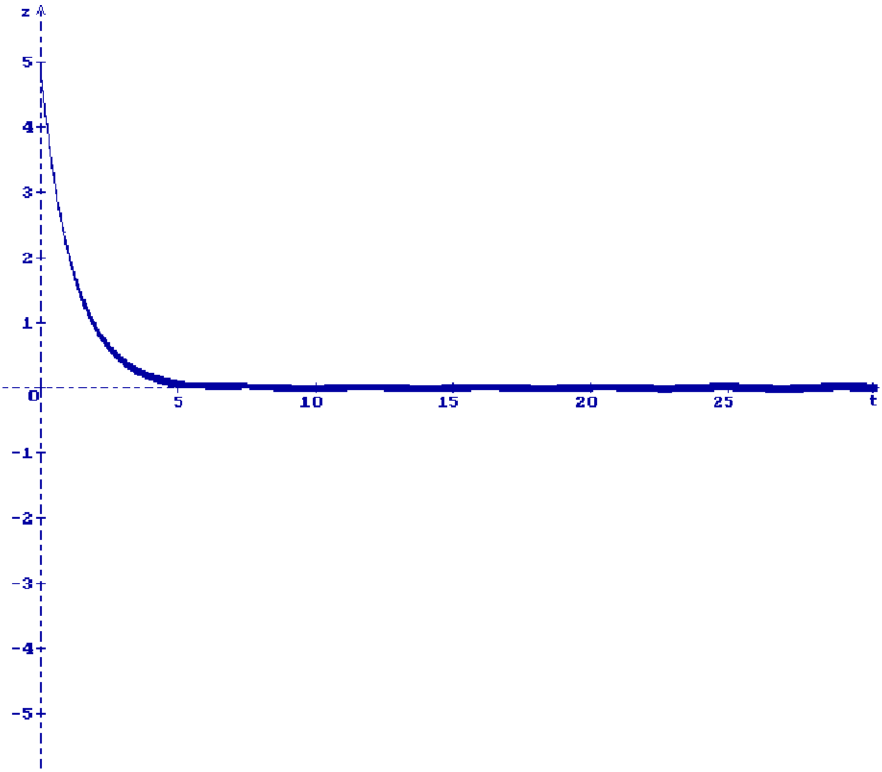


Fig. 10.18 Bounded solution of (10.73) in the point $L(87.5, 85)$, $a_0 = 5$

As was shown in [227, 229], if the order of nonlinearity of a considered system is higher than 1 then sufficient conditions for asymptotic mean square stability of the trivial solution of the linear part of a considered nonlinear system at the same time are sufficient conditions for stability in probability of the trivial solution of the initial nonlinear system. So, the conditions of Theorem 10.3 are sufficient conditions for the stability in probability of the trivial solution of the nonlinear system (10.88) and, respectively, for the stability in probability of the equilibrium point of the system (10.87).

10.4.2 Stability of the Difference Analogue

Consider the difference analogues of the systems (10.87)

$$\begin{aligned} x_1(i+1) &= x_1(i) + \Delta x_1(i)\big(a - a_1 x_1(i) - a_2 x_2(i)\big) + \sqrt{\Delta}\sigma_1\big(x_1(i) - x_1^*\big)\xi_1(i+1), \\ x_2(i+1) &= (1 - \Delta b)x_2(i) + \Delta b_1 x_1(i - k_1)x_2(i - k_2) \\ &\quad + \sqrt{\Delta}\sigma_2\big(x_2(i) - x_2^*\big)\xi_2(i+1) \end{aligned} \tag{10.92}$$

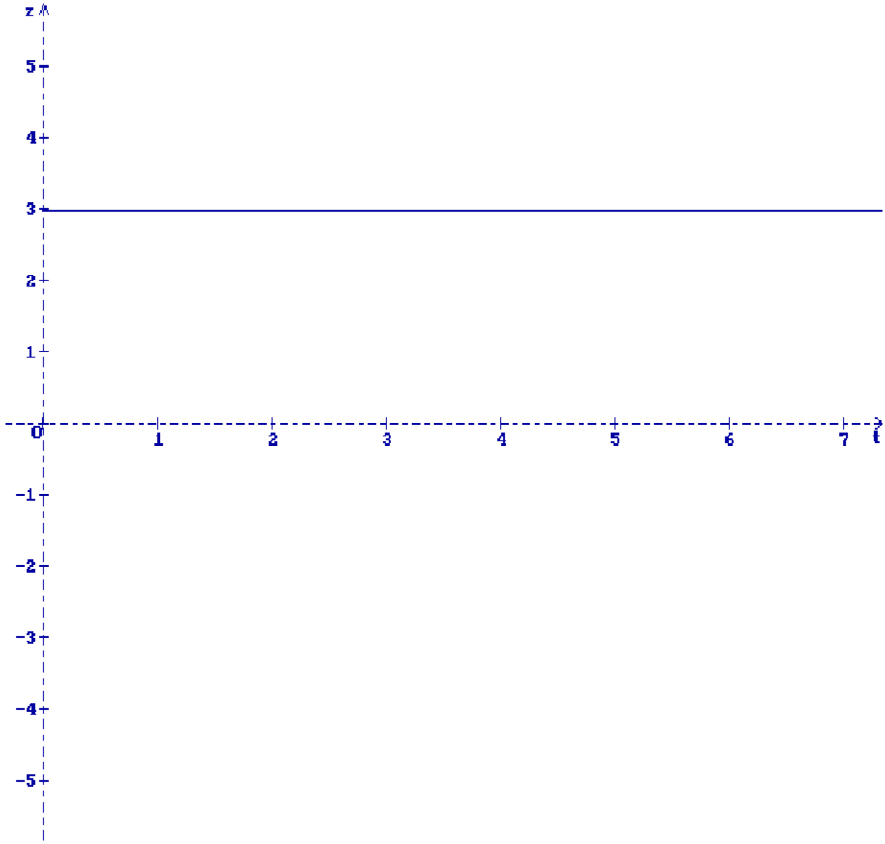


Fig. 10.19 Bounded solution of (10.73) in the point $M(40,40)$, $a_0=3$

and (10.88):

$$\begin{aligned} z_1(i+1) &= \big(1-\Delta a_1 x_1^*\big)z_1(i) - \Delta a_2 x_1^* z_2(i) + \sqrt{\Delta}\sigma_1 z_1(i)\xi_1(i+1), \\ z_2(i+1) &= (1-\Delta b)z_2(i) + \Delta b_1\big(x_2^* z_1(i-k_1) + x_1^* z_2(i-k_2)\big) \\ &\quad + \sqrt{\Delta}\sigma_2 z_2(i)\xi_2(i+1). \end{aligned} \tag{10.93}$$

Here $k_m = h_m/\Delta$ is an integer, Δ is the step of discretization, $x_m(i)=x_m(t_i)$, $t_i = i\Delta$, $\xi_m(i+1) = (w_m(t_{i+1}) - w_m(t_i))/\sqrt{\Delta}$, $m=1,2$, $i \in Z$, $\xi_1(i)$ and $\xi_2(i)$, $i \in Z$, are independent of each other $\mathfrak{F}_i$-adapted random variables such that $\mathbf{E}\xi_m(i)=0$, $\mathbf{E}\xi_m^2(i)=1$, $m=1,2$.

To get sufficient conditions for asymptotic mean square stability of the trivial solution of the system (10.93) let us construct an appropriate Lyapunov functional. Following the procedure of the construction of the Lyapunov functionals, rewrite

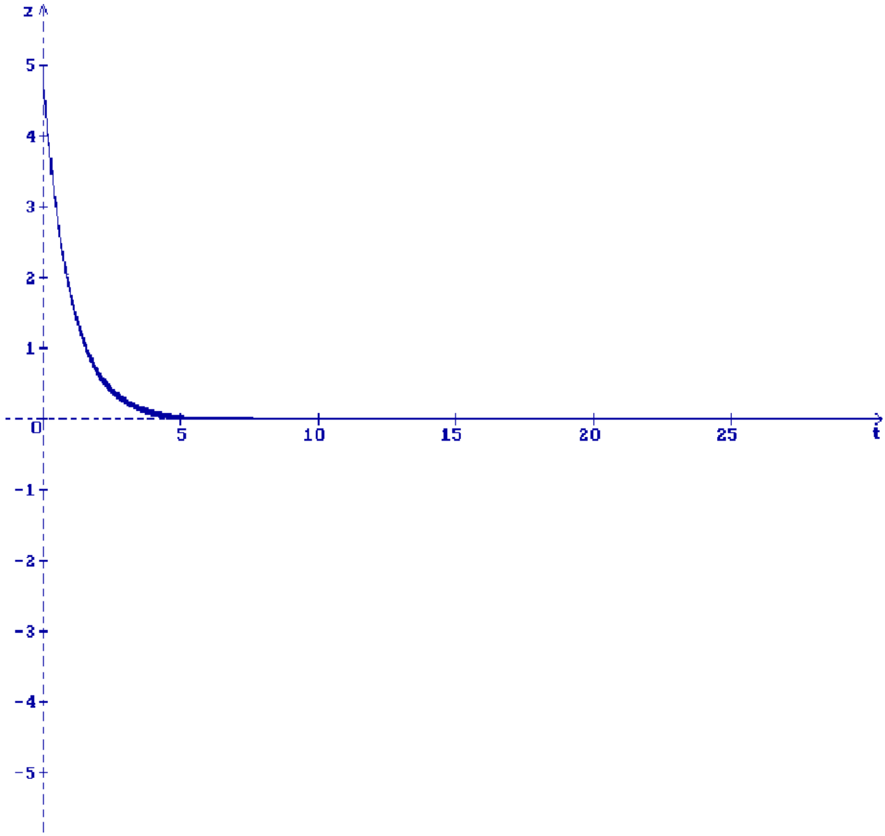


Fig. 10.20 Stable solution of (10.73) in the point $L_1(88, 85)$, $a_0 = 5$

the second equation of the system (10.93) in the form

$$z_2(i+1) = z_2(i) + \Delta b_1 x_2^* z_1(i) - \Delta b_1 \big(x_2^* \bar{\Delta} F_1(i) + x_1^* \bar{\Delta} F_2(i)\big) + \sqrt{\Delta}\sigma_2 z_2(i)\xi_2(i+1),$$

where $\bar{\Delta} F_m(i) = F_m(i+1) - F_m(i)$,

$$F_m(i) = \sum_{j=1}^{k_m} z_m(i-j), \quad m = 1, 2,$$

and put

$$Z_2(i) = z_2(i) + \Delta b_1 \big(x_2^* F_1(i) + x_1^* F_2(i)\big).$$

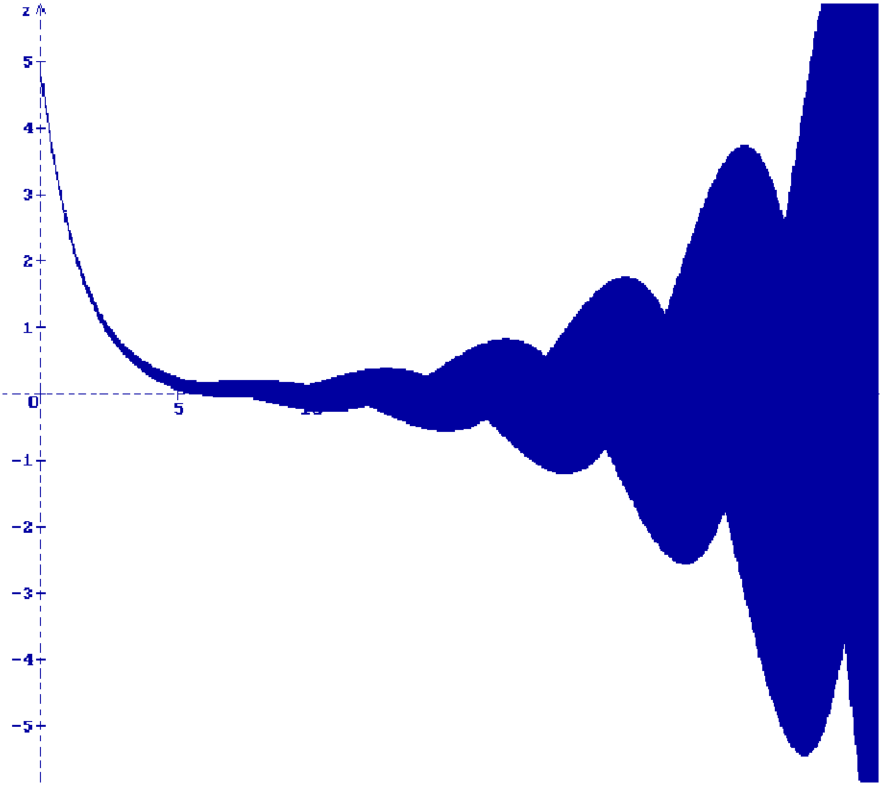


Fig. 10.21 Unstable solution of (10.73) in the point $L_2(87,85)$, $a_0=5$

Then the system (10.93) can be represented by

$$\begin{aligned} z_1(i+1) &= \big(1-\Delta a_1 x_1^*\big)z_1(i) - \Delta a_2 x_1^* z_2(i) + \sqrt{\Delta}\sigma_1 z_1(i)\xi_1(i+1), \\ Z_2(i+1) &= \Delta b_1 x_2^* z_1(i) + Z_2(i) + \sqrt{\Delta}\sigma_2 z_2(i)\xi_2(i+1). \end{aligned} \tag{10.94}$$

A Lyapunov functional $V(i)$ for the system (10.94) is constructed as $V(i) = V_1(i) + V_2(i)$, where

$$V_1(i) = z_1^2(i) + 2\mu z_1(i) Z_2(i) + \gamma Z_2^2(i),$$

and $V_2(i)$ is chosen by the standard way after estimation of $\mathbf{E}\bar{\Delta} V_1(i)$.

It is easy to see that

$$\mathbf{E}\bar{\Delta} V_1(i) = \mathbf{E}\big(V_1(i+1) - V_1(i)\big) = J_1 + J_2 + J_3,$$

where

$$J_1 = \mathbf{E}\big[\big(\big(1-\Delta a_1 x_1^*\big)z_1(i) - \Delta a_2 x_1^* z_2(i) + \sqrt{\Delta}\sigma_1 z_1(i)\xi_1(i+1)\big)^2 - z_1^2(i)\big],$$

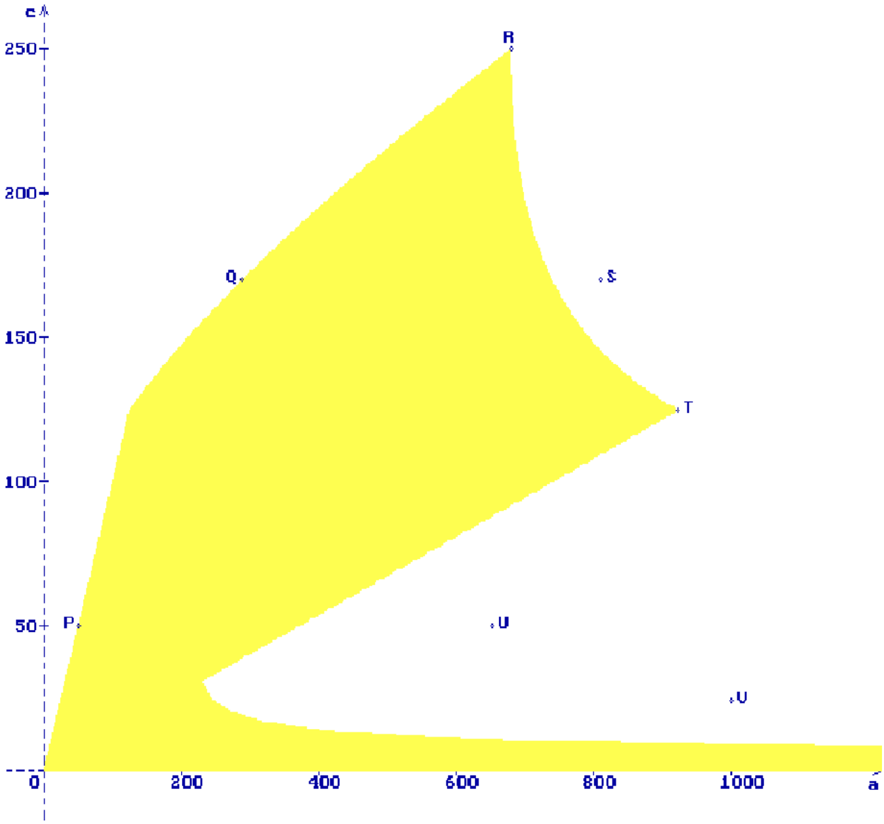


Fig. 10.22 Region of sufficient stability condition for (10.73): $p=0$, $h=0.024$, $\Delta=0.008$

$$J_2 = 2\mu\mathbf{E}\Big[\big(\big(1-\Delta a_1x_1^*\big)z_1(i)-\Delta a_2x_1^*z_2(i)+\sqrt{\Delta}\sigma_1z_1(i)\xi_1(i+1)\big)$$
$$\times\big(\Delta b_1x_2^*z_1(i)+Z_2(i)+\sqrt{\Delta}\sigma_2z_2(i)\xi_2(i+1)\big)-z_1(i)Z_2(i)\Big],$$
$$J_3 = \gamma\mathbf{E}\Big[\big(\Delta b_1x_2^*z_1(i)+Z_2(i)+\sqrt{\Delta}\sigma_2z_2(i)\xi_2(i+1)\big)^2-Z_2^2(i)\Big].$$

Using the condition $\mathbf{E}\xi_m(i+1)=0$, $m=1,2$, transform J_1 as

$$J_1 = \big[\big(1-\Delta a_1x_1^*\big)^2-1\big]\mathbf{E}z_1^2(i)+\Delta^2a_2^2\big(x_1^*\big)^2\mathbf{E}z_2^2(i)$$
$$+\Delta\sigma_1^2\mathbf{E}z_1^2(i)-2\big(1-\Delta a_1x_1^*\big)\Delta a_2x_1^*\mathbf{E}z_1(i)z_2(i)$$
$$=\Delta\Big[\big(-a_1x_1^*\big(2-\Delta a_1x_1^*\big)+\sigma_1^2\big)\mathbf{E}z_1^2(i)+\Delta a_2^2\big(x_1^*\big)^2\mathbf{E}z_2^2(i)$$
$$-2\big(1-\Delta a_1x_1^*\big)a_2x_1^*\mathbf{E}z_1(i)z_2(i)\Big].$$

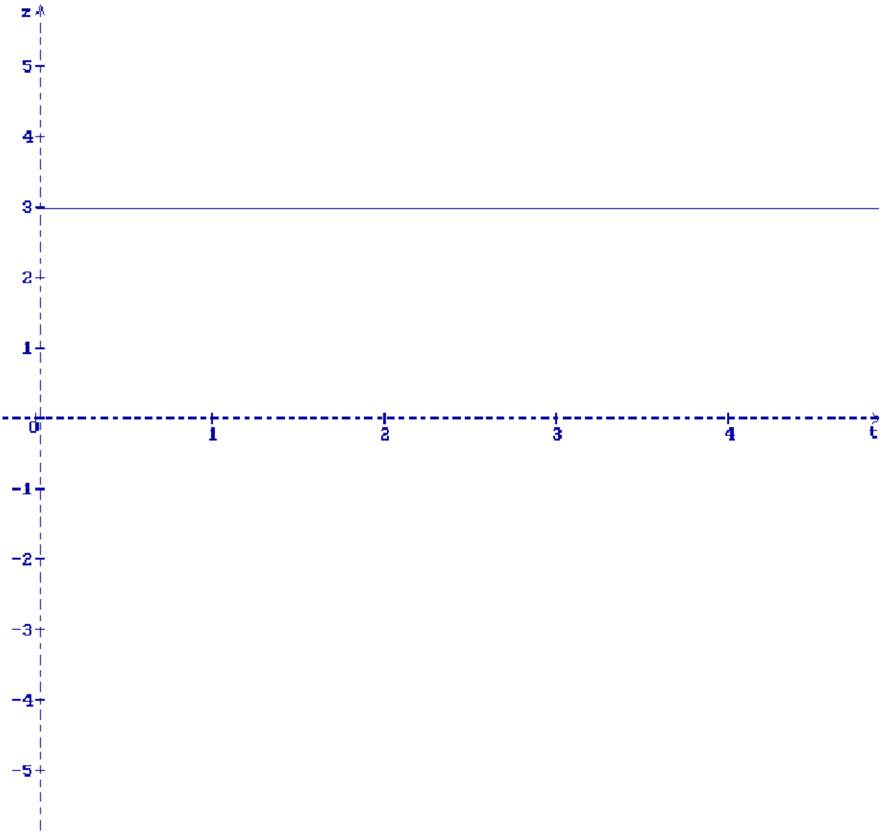


Fig. 10.23 Bounded solution of (10.73) in the point $P(50, 50)$, $a_0 = 3$

Analogously we have

$$\begin{aligned}
J_2 &= 2\mu\mathbf{E}\big[\big((1-\Delta a_1 x_1^*)z_1(i) - \Delta a_2 x_1^* z_2(i)\big)\big(\Delta b_1 x_2^* z_1(i) + Z_2(i)\big) - z_1(i)Z_2(i)\big] \\
&= 2\mu\big[(1-\Delta a_1 x_1^*)\Delta b_1 x_2^*\mathbf{E}z_1^2(i) - \Delta^2 Ab\mathbf{E}z_1(i)z_2(i) \\
&\quad - \Delta a_1 x_1^*\mathbf{E}z_1(i)Z_2(i) - \Delta a_2 x_1^*\mathbf{E}z_2(i)Z_2(i)\big] \\
&= 2\mu\Delta\big[(1-\Delta a_1 x_1^*)b_1 x_2^*\mathbf{E}z_1^2(i) - \Delta Ab\mathbf{E}z_1(i)z_2(i) \\
&\quad - a_1\mathbf{E}z_1(i)\big(x_1^* z_2(i) + \Delta b\big(x_2^* F_1(i) + x_1^* F_2(i)\big)\big) \\
&\quad - a_2\mathbf{E}z_2(i)\big(x_1^* z_2(i) + \Delta b\big(x_2^* F_1(i) + x_1^* F_2(i)\big)\big)\big] \\
&= 2\mu\Delta\big[(1-\Delta a_1 x_1^*)b_1 x_2^*\mathbf{E}z_1^2(i) - \big(\Delta Ab + a_1 x_1^*\big)\mathbf{E}z_1(i)z_2(i) \\
&\quad - \Delta a_1 b\big(x_2^*\mathbf{E}z_1(i)F_1(i) + x_1^*\mathbf{E}z_1(i)F_2(i)\big) \\
&\quad - a_2 x_1^*\mathbf{E}z_2^2(i) - \Delta b\big(A\mathbf{E}z_2(i)F_1(i) + a_2 x_1^*\mathbf{E}z_2(i)F_2(i)\big)\big]
\end{aligned}$$

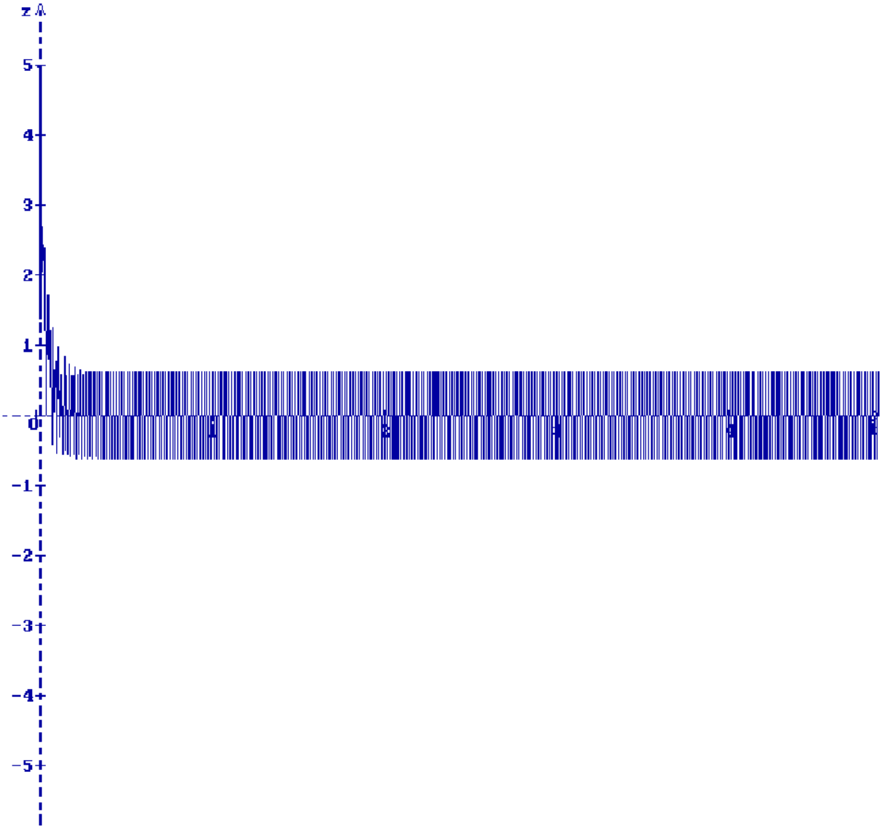


Fig. 10.24 Bounded solution of (10.73) in the point $Q(288.65, 170)$, $a_0 = 5$

and

$$
\begin{aligned}
J_3 &= \gamma\Delta\big[\Delta b_1^2\big(x_2^*\big)^2\mathbf{E}z_1^2(i) + \sigma_2^2\mathbf{E}z_2^2(i) \\
&\quad + 2b_1x_2^*\mathbf{E}z_1(i)\big(z_2(i) + \Delta b_1\big(x_2^*F_1(i) + x_1^*F_2(i)\big)\big)\big] \\
&= \gamma\Delta\big[\Delta b_1^2\big(x_2^*\big)^2\mathbf{E}z_1^2(i) + \sigma_2^2\mathbf{E}z_2^2(i) + 2b_1x_2^*\mathbf{E}z_1(i)z_2(i) \\
&\quad + 2\Delta b_1^2x_2^*\big(x_2^*\mathbf{E}z_1(i)F_1(i) + x_1^*\mathbf{E}z_1(i)F_2(i)\big)\big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}\bar{\Delta}V_1(i) = \Delta\big[&\big(-a_1x_1^*\big(2 - \Delta a_1x_1^*\big) + \sigma_1^2\big)\mathbf{E}z_1^2(i) + \Delta a_2^2\big(x_1^*\big)^2\mathbf{E}z_2^2(i) \\
&- 2\big(1 - \Delta a_1x_1^*\big)a_2x_1^*\mathbf{E}z_1(i)z_2(i)\big] \\
&+ 2\mu\Delta\big[\big(1 - \Delta a_1x_1^*\big)b_1x_2^*\mathbf{E}z_1^2(i) - \big(\Delta Ab + a_1x_1^*\big)\mathbf{E}z_1(i)z_2(i)
\end{aligned}
$$

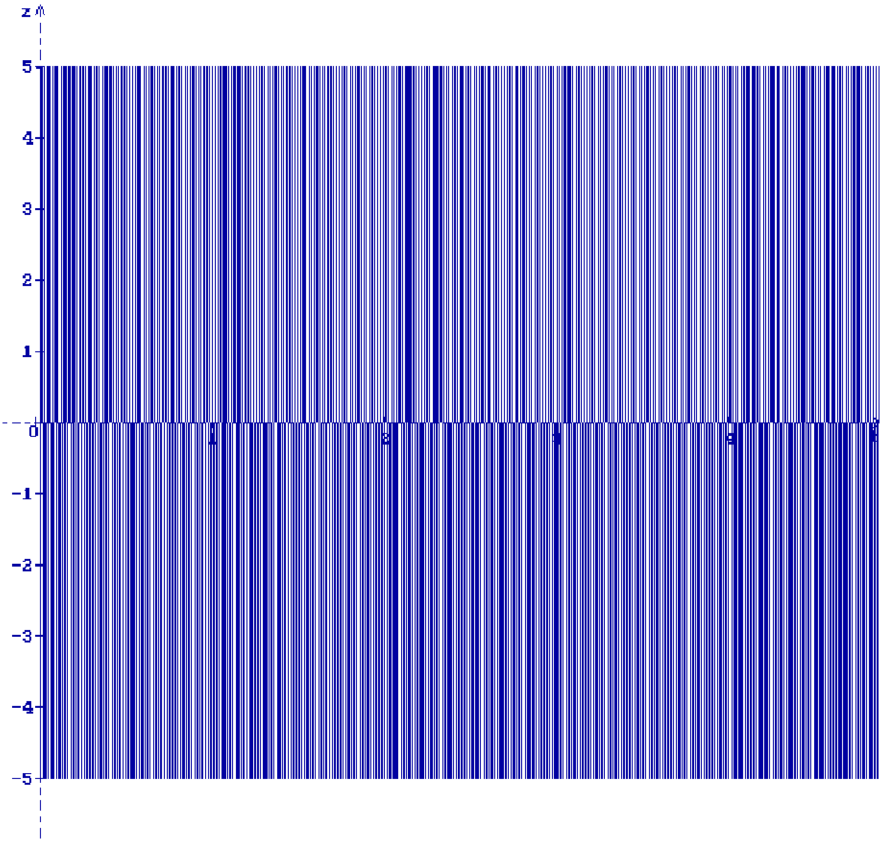


Fig. 10.25 Bounded solution of (10.73) in the point $R(680, 250.079)$, $a_0 = 5$

$$
\begin{aligned}
&- \Delta a_1 b\big(x_2^*\mathbf{E}z_1(i)F_1(i) + x_1^*\mathbf{E}z_1(i)F_2(i)\big) \\
&- a_2 x_1^*\mathbf{E}z_2^2(i) - \Delta a_2 b\big(x_2^*\mathbf{E}z_2(i)F_1(i) + x_1^*\mathbf{E}z_2(i)F_2(i)\big)\big] \\
&+ \gamma\Delta\big[\Delta b_1^2\big(x_2^*\big)^2\mathbf{E}z_1^2(i) + \sigma_2^2\mathbf{E}z_2^2(i) + 2b_1x_2^*\mathbf{E}z_1(i)z_2(i) \\
&+ 2\Delta b_1^2 x_2^*\big(x_2^*\mathbf{E}z_1(i)F_1(i) + x_1^*\mathbf{E}z_1(i)F_2(i)\big)\big] \\
= {}&\Delta\big[-a_1x_1^*\big(2 - \Delta a_1 x_1^*\big) + \sigma_1^2 \\
&+ 2\mu b_1 x_2^*\big(1 - \Delta a_1 x_1^*\big) + \gamma\Delta b_1^2\big(x_2^*\big)^2\big]\mathbf{E}z_1^2(i) \\
&+ \Delta\big[\Delta a_2^2\big(x_1^*\big)^2 - 2\mu a_2 x_1^* + \gamma\sigma_2^2\big]\mathbf{E}z_2^2(i) \\
&- 2\Delta\big[a_2x_1^*\big(1 - \Delta a_1 x_1^*\big) + \mu\big(\Delta Ab + a_1x_1^*\big) - \gamma b_1 x_2^*\big]\mathbf{E}z_1(i)z_2(i)
\end{aligned}
$$

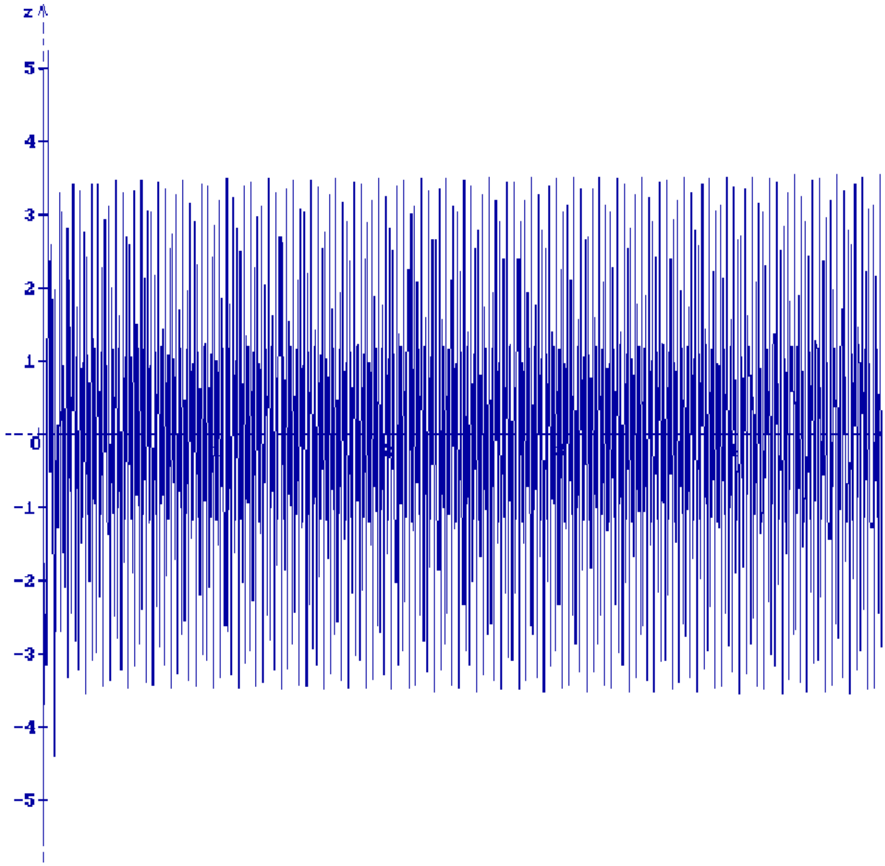


Fig. 10.26 Bounded solution of (10.73) in the point $S(810, 170)$, $a_0 = 5$

$$
\begin{aligned}
&+ 2\Delta^2 b_1 x_2^* \big[\gamma b_1 x_2^* - \mu a_1 x_1^*\big] \mathbf{E} z_1(i) F_1(i) \\
&+ 2\Delta^2 b \big[\gamma b_1 x_2^* - \mu a_1 x_1^*\big] \mathbf{E} z_1(i) F_2(i) \\
&- 2\Delta^2 \mu A b \mathbf{E} z_2(i) F_1(i) - 2\Delta^2 \mu a_2 b x_1^* \mathbf{E} z_2(i) F_2(i). \qquad (10.95)
\end{aligned}
$$

Note that

$$
\begin{aligned}
2\Delta z_m(i) F_m(i) &= 2\Delta \sum_{j=1}^{k_m} z_m(i) z_m(i-j) \le \Delta \sum_{j=1}^{k_m} \big(z_m^2(i) + z_m^2(i-j)\big) \\
&= h_m z_m^2(i) + \Delta \sum_{j=1}^{k_m} z_m^2(i-j), \quad m = 1, 2, \qquad (10.96)
\end{aligned}
$$

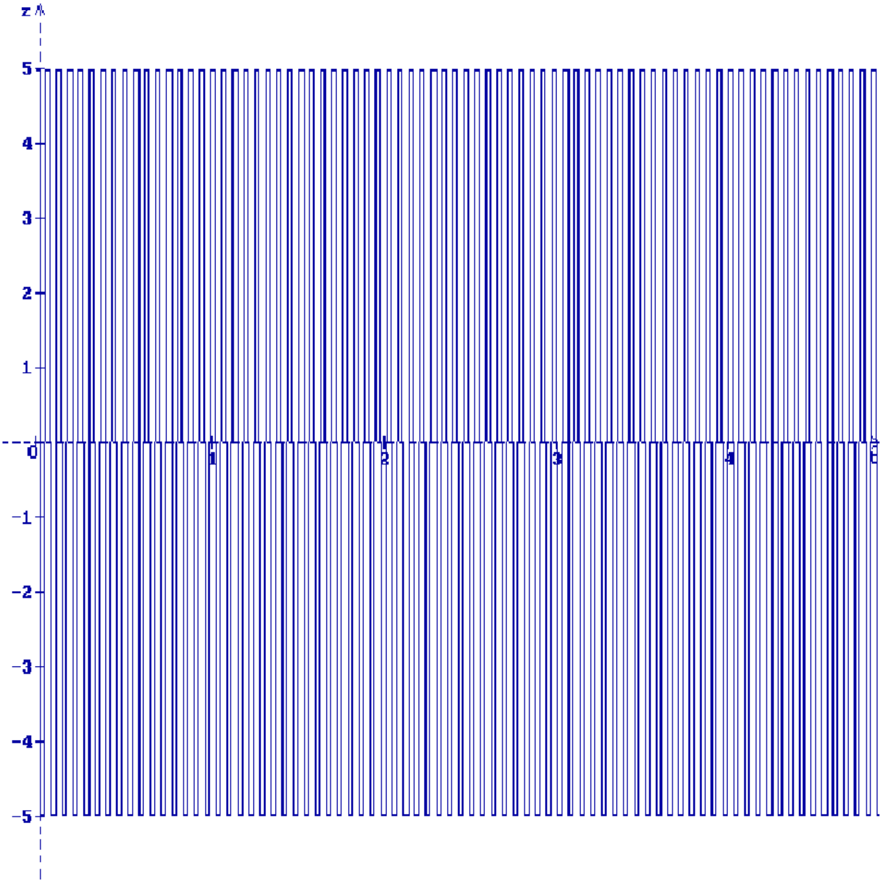


Fig. 10.27 Bounded solution of (10.73) in the point $T(923.63, 125)$, $a_0 = 5$

and for arbitrary positive γ_1, γ_2

$$2\Delta z_1(i)F_2(i) = 2\Delta \sum_{j=1}^{k_2} z_1(i)z_2(i-j) \leq \Delta \sum_{j=1}^{k_2} \big(\gamma_1^{-1} z_1^2(i) + \gamma_1 z_2^2(i-j)\big)$$

$$= \gamma_1^{-1} h_2 z_1^2(i) + \gamma_1 \Delta \sum_{j=1}^{k_2} z_2^2(i-j), \tag{10.97}$$

$$2\Delta z_2(i)F_1(i) = 2\Delta \sum_{j=1}^{k_1} z_2(i)z_1(i-j) \leq \Delta \sum_{j=1}^{k_1} \big(\gamma_2 z_2^2(i) + \gamma_2^{-1} z_1^2(i-j)\big)$$

$$= \gamma_2 h_1 z_2^2(i) + \gamma_2^{-1} \Delta \sum_{j=1}^{k_1} z_1^2(i-j). \tag{10.98}$$

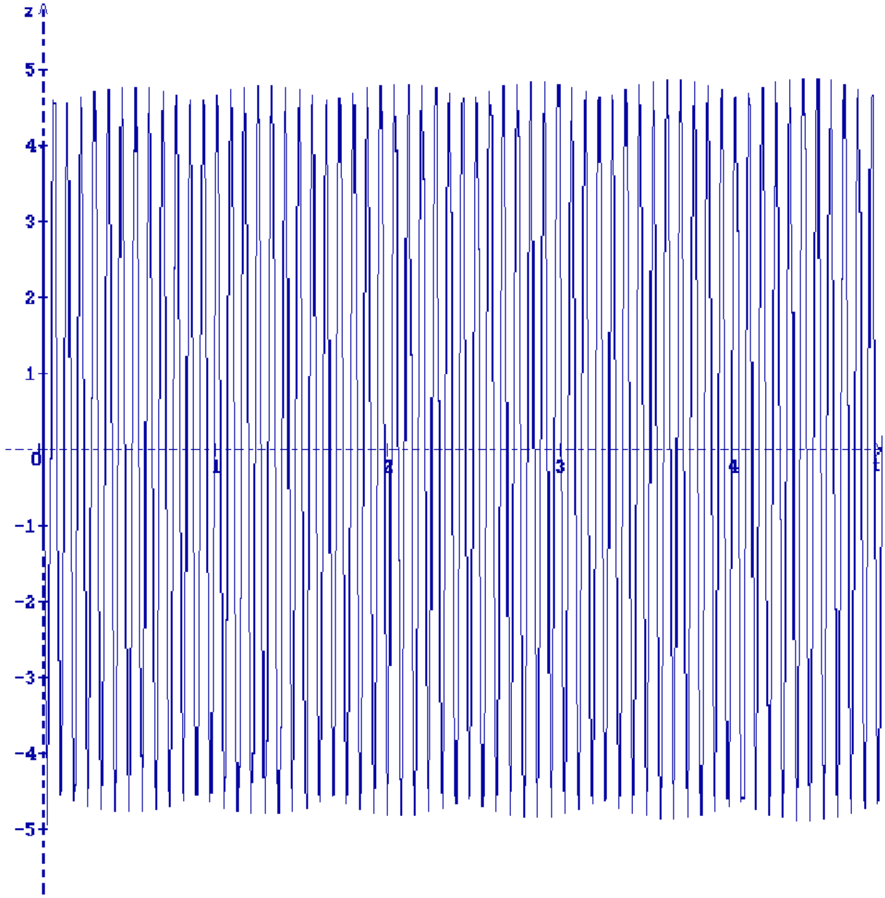


Fig. 10.28 Bounded solution of (10.73) in the point $U(652.6, 50)$, $a_0 = 4$

Substituting the estimations (10.96)–(10.98) into (10.95) we obtain

$$\begin{aligned}
\mathbf{E}\bar{\Delta}V_1(i) \le{}& \Delta\big[-a_1x_1^*\big(2-\Delta a_1x_1^*\big)+\sigma_1^2\\
&+2\mu b_1x_2^*\big(1-\Delta a_1x_1^*\big)+\gamma\Delta b_1^2\big(x_2^*\big)^2\big]\mathbf{E}z_1^2(i)\\
&+\Delta\big[\Delta a_2^2\big(x_1^*\big)^2-2\mu a_2x_1^*+\gamma\sigma_2^2\big]\mathbf{E}z_2^2(i)\\
&-2\Delta\big[a_2x_1^*\big(1-\Delta a_1x_1^*\big)+\mu\big(\Delta Ab+a_1x_1^*\big)-\gamma b_1x_2^*\big]\mathbf{E}z_1(i)z_2(i)\\
&+\Delta b_1x_2^*\big|\gamma b_1x_2^*-\mu a_1x_1^*\big|\left(h_1\mathbf{E}z_1^2(i)+\Delta\sum_{j=1}^{k_1}\mathbf{E}z_1^2(i-j)\right)
\end{aligned}$$

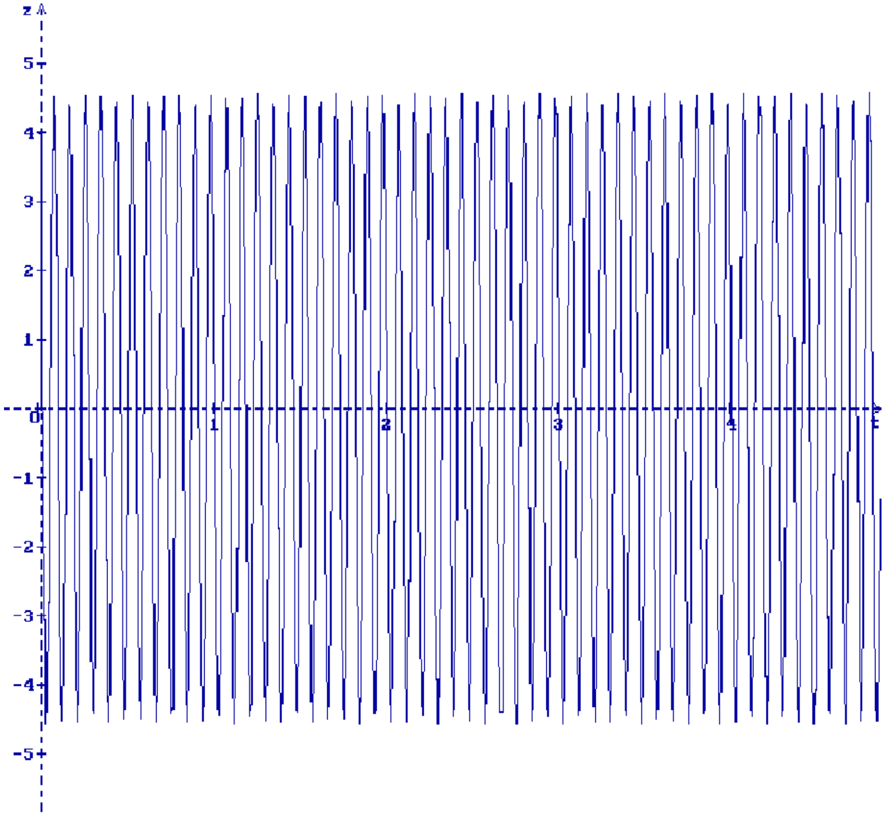


Fig. 10.29 Bounded solution of (10.73) in the point $V(1000, 24.16)$, $a_0 = 4$

$$
\begin{aligned}
&+ \Delta b\left|\gamma b_1 x_2^* - \mu a_1 x_1^*\right|\left(\gamma_1^{-1} h_2 \mathbf{E} z_1^2(i) + \gamma_1 \Delta \sum_{j=1}^{k_2} \mathbf{E} z_2^2(i-j)\right) \\
&+ \Delta \mu A b\left(\gamma_2 h_1 \mathbf{E} z_2^2(i) + \gamma_2^{-1} \Delta \sum_{j=1}^{k_1} \mathbf{E} z_1^2(i-j)\right) \\
&+ \Delta \mu a_2 b x_1^*\left(h_2 \mathbf{E} z_2^2(i) + \Delta \sum_{j=1}^{k_2} \mathbf{E} z_2^2(i-j)\right) \\
= {} & \Delta\big[-a_1 x_1^*\big(2 - \Delta a_1 x_1^*\big) + \sigma_1^2 + 2\mu b_1 x_2^*\big(1 - \Delta a_1 x_1^*\big) + \gamma \Delta b_1^2\big(x_2^*\big)^2 \\
&+ b_1\left|\gamma b_1 x_2^* - \mu a_1 x_1^*\right|\big(x_2^* h_1 + \gamma_1^{-1} x_1^* h_2\big)\big]\mathbf{E} z_1^2(i) \\
&+ \Delta\big[\Delta a_2^2\big(x_1^*\big)^2 - 2\mu a_2 x_1^* + \gamma \sigma_2^2 + \mu b\big(\gamma_2 A h_1 + a_2 x_1^* h_2\big)\big]\mathbf{E} z_2^2(i)
\end{aligned}
$$

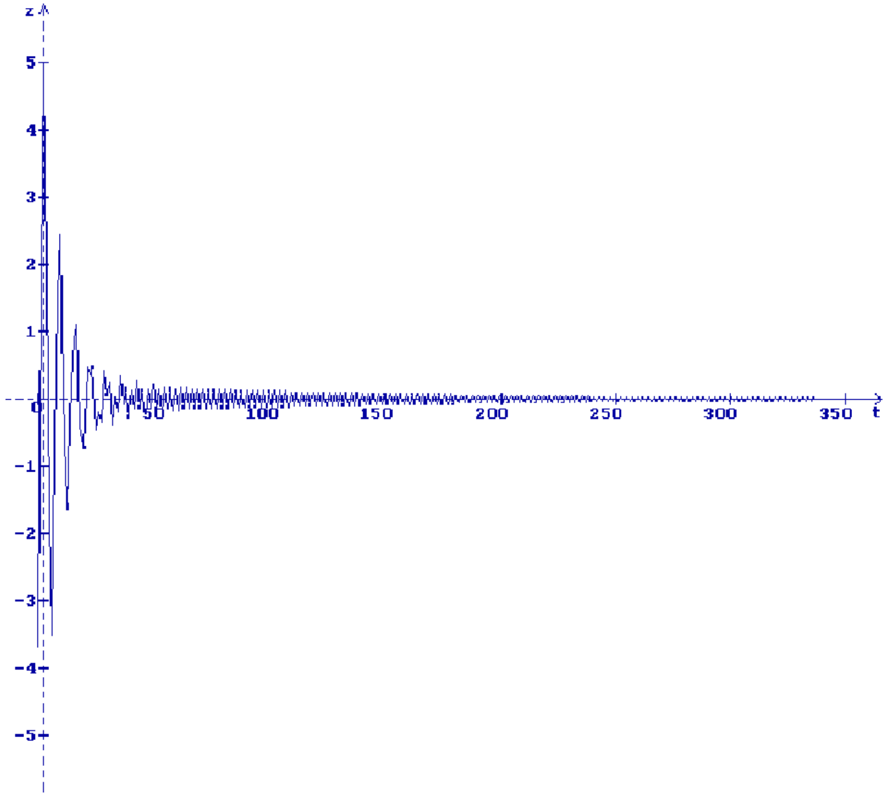


Fig. 10.30 Stable solution of (10.73) in the point $A_0(5.2, 1)$, $p = 0$, $h = 2.4$, $\Delta = 1.2$, $a_0 = 5$

$$- 2\Delta\big[a_2x_1^*\big(1 - \Delta a_1x_1^*\big) + \mu\big(\Delta Ab + a_1x_1^*\big) - \gamma b_1x_2^*\big]\mathbf{E}z_1(i)z_2(i)$$
$$+ \Delta^2\left(q_1\sum_{j=1}^{k_1}\mathbf{E}z_1^2(i-j) + q_2\sum_{j=1}^{k_2}\mathbf{E}z_2^2(i-j)\right),$$

where

$$\begin{aligned} q_1 &= b_1\big[x_2^*\big|\gamma b_1x_2^* - \mu a_1x_1^*\big| + \gamma_2^{-1}\mu Ax_1^*\big], \\ q_2 &= b\big[\gamma_1\big|\gamma b_1x_2^* - \mu a_1x_1^*\big| + \mu a_2x_1^*\big]. \end{aligned} \tag{10.99}$$

Following the procedure of the construction of the Lyapunov functionals choose the auxiliary part $V_2(i)$ of the functional $V(i)$ in the form

$$V_2(i) = \Delta^2\sum_{m=1}^{2}q_m\sum_{j=1}^{k_m}(k_m - j + 1)z_m^2(i-j).$$

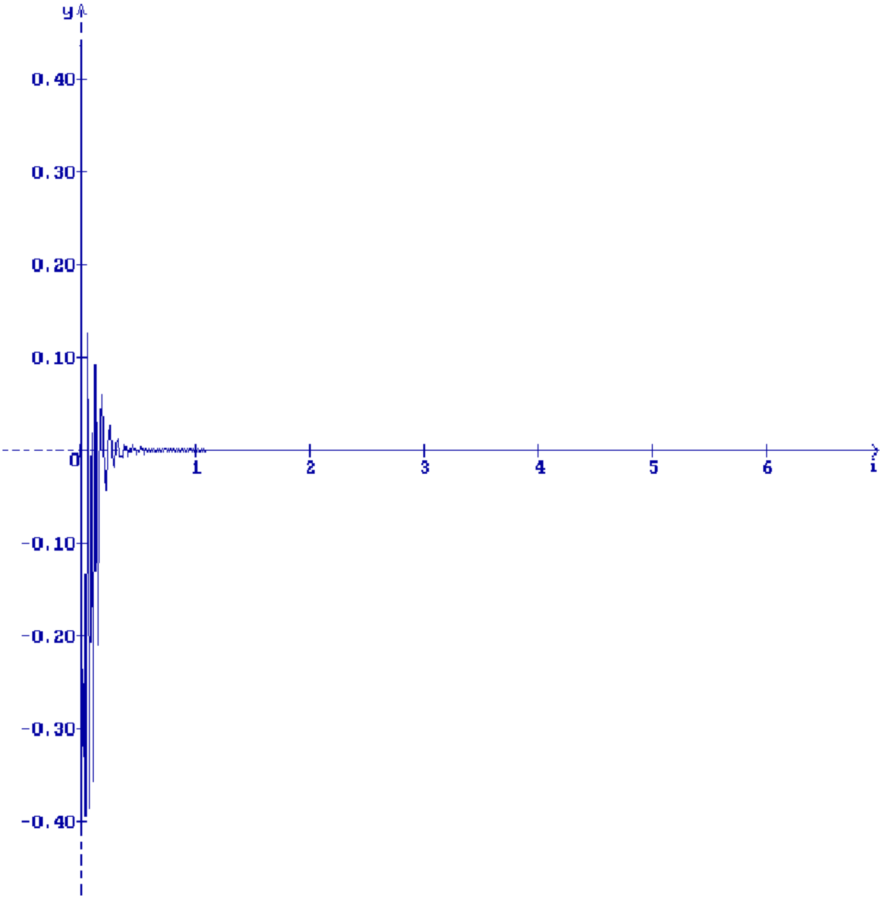


Fig. 10.31 Stable solution of (10.72) in the point $A(520, 100)$, $a_0 = 0.437$

Then

$$\bar{\Delta} V_2(i) = \Delta^2 \sum_{m=1}^{2} q_m \left(k_m z_m^2(i) - \sum_{j=1}^{k_m} z_m^2(i-j) \right).$$

So, via (10.99) for the functional $V(i) = V_1(i) + V_2(i)$ we obtain

$$\begin{aligned}\mathbf{E}\bar{\Delta} V(i) \le \Delta\big[&-a_1 x_1^*\big(2 - \Delta a_1 x_1^*\big) + \sigma_1^2 + 2\mu b_1 x_2^*\big(1 - \Delta a_1 x_1^*\big) + \gamma \Delta b_1^2 \big(x_2^*\big)^2 \\ &+ b_1\big|\gamma b_1 x_2^* - \mu a_1 x_1^*\big|\big(x_2^* h_1 + \gamma_1^{-1} x_1^* h_2\big) \\ &+ b_1 h_1\big(x_2^*\big|\gamma b_1 x_2^* - \mu a_1 x_1^*\big| + \gamma_2^{-1}\mu A x_1^*\big)\big]\mathbf{E} z_1^2(i) \\ &+ \Delta\big[\Delta a_2^2\big(x_1^*\big)^2 - 2\mu a_2 x_1^* + \gamma \sigma_2^2 + \mu b\big(\gamma_2 A h_1 + a_2 x_1^* h_2\big)\end{aligned}$$

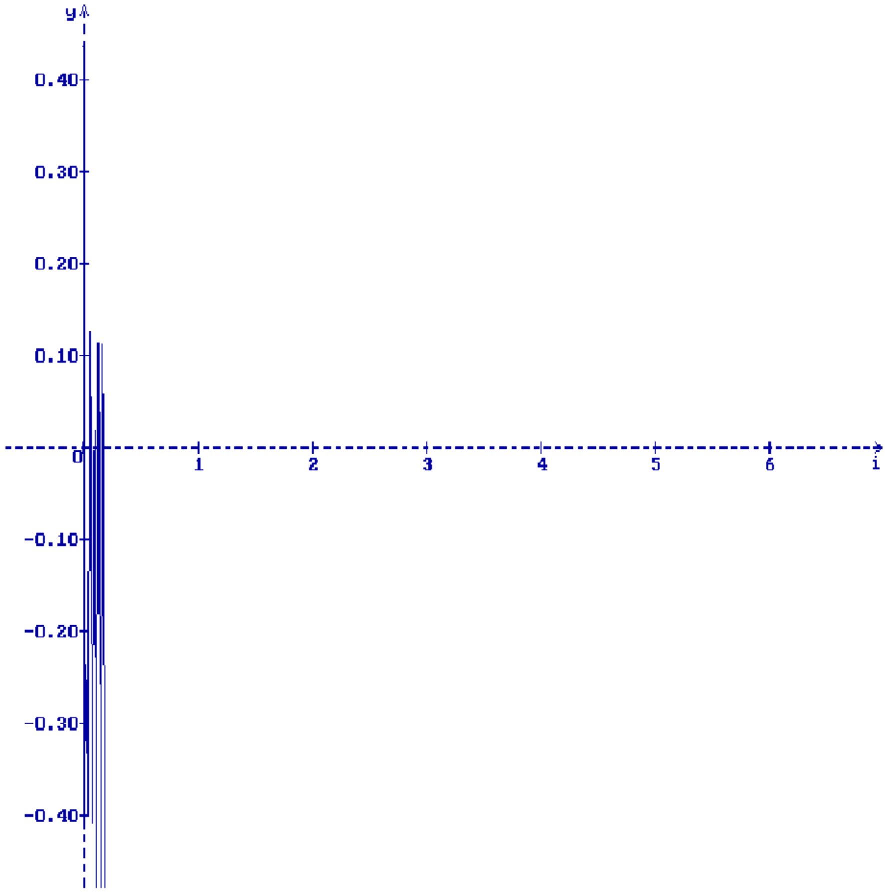


Fig. 10.32 Unstable solution of (10.72) in the point $A(520, 100)$, $a_0 = 0.438$

$$\begin{aligned}
&+ bh_2\big(\gamma_1\big|\gamma b_1 x_2^* - \mu a_1 x_1^*\big| + \mu a_2 x_1^*\big)\big]\mathbf{E}z_2^2(i)\\
&- 2\Delta\big[a_2 x_1^*\big(1 - \Delta a_1 x_1^*\big) + \mu\big(\Delta Ab + a_1 x_1^*\big) - \gamma b_1 x_2^*\big]\mathbf{E}z_1(i)z_2(i)\\
= {}&\Delta\big[-a_1 x_1^*\big(2 - \Delta a_1 x_1^*\big) + \sigma_1^2 + 2\mu b_1 x_2^*\big(1 - \Delta a_1 x_1^*\big) + \gamma\Delta b_1^2\big(x_2^*\big)^2\\
&+ b_1\big|\gamma b_1 x_2^* - \mu a_1 x_1^*\big|\big(2x_2^* h_1 + \gamma_1^{-1} x_1^* h_2\big) + \gamma_2^{-1}\mu Abh_1\big]\mathbf{E}z_1^2(i)\\
&+ \Delta\big[\Delta a_2^2\big(x_1^*\big)^2 - 2\mu a_2 x_1^* + \gamma\sigma_2^2 + \mu b\big(\gamma_2 Ah_1 + a_2 x_1^* h_2\big)\\
&+ bh_2\big(\gamma_1\big|\gamma b_1 x_2^* - \mu a_1 x_1^*\big| + \mu a_2 x_1^*\big)\big]\mathbf{E}z_2^2(i)\\
&+ 2\Delta\big[\gamma b_1 x_2^* - \mu a_1 x_1^* - a_2 x_1^*\big(1 - \Delta a_1 x_1^*\big) - \Delta\mu Ab\big]\mathbf{E}z_1(i)z_2(i).
\end{aligned}$$

As a result we find that the functional $V(i)$ satisfies the condition

$$\mathbf{E}\bar{\Delta}V(i) \le \mathbf{E}z'(i)Pz(i),$$

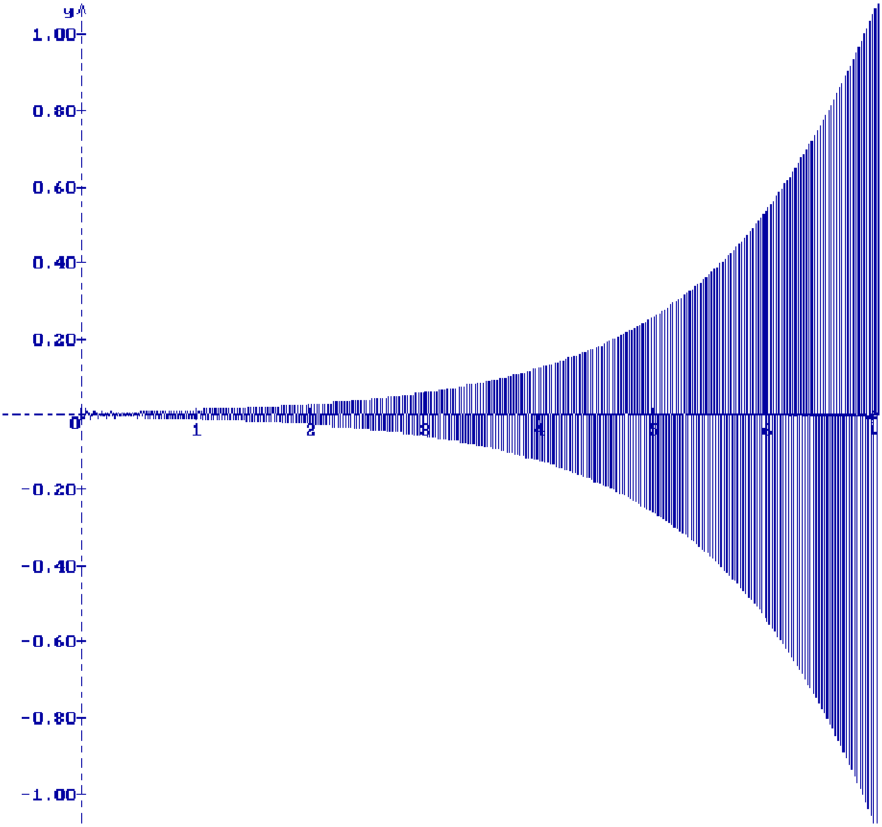


Fig. 10.33 Unstable solution of (10.72) in the point $C(540, 100)$, $a_0 = 0.001$

where

$$z(i) = \begin{pmatrix} z_1(i) \\ z_2(i) \end{pmatrix}, \qquad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix},$$

$$\begin{aligned} p_{11} &= \Delta\big[-a_1x_1^*(2-\Delta a_1x_1^*) + \sigma_1^2 + 2\mu b_1x_2^*(1-\Delta a_1x_1^*) + \gamma\Delta b_1^2(x_2^*)^2 \\ &\quad + b_1|\gamma b_1x_2^* - \mu a_1x_1^*|(2x_2^*h_1 + \gamma_1^{-1}x_1^*h_2) + \gamma_2^{-1}\mu Abh_1\big], \\ p_{22} &= \Delta\big[\Delta a_2^2(x_1^*)^2 - 2\mu a_2x_1^* + \gamma\sigma_2^2 + \mu b(\gamma_2Ah_1 + a_2x_1^*h_2) \\ &\quad + bh_2(\gamma_1|\gamma b_1x_2^* - \mu a_1x_1^*| + \mu a_2x_1^*)\big], \\ p_{12} &= 2\Delta\big[\gamma b_1x_2^* - \mu a_1x_1^* - a_2x_1^*(1-\Delta a_1x_1^*) - \Delta\mu Ab\big]. \end{aligned} \tag{10.100}$$

From Theorem 1.1 we have the following.

Corollary 10.3 *If there exist numbers* μ, $\gamma > \mu^2$, $\gamma_1 > 0$, $\gamma_2 > 0$, *such that the symmetric matrix* P *with elements* (10.100) *is a negative definite one*, *i.e.*, $p_{11} < 0$,

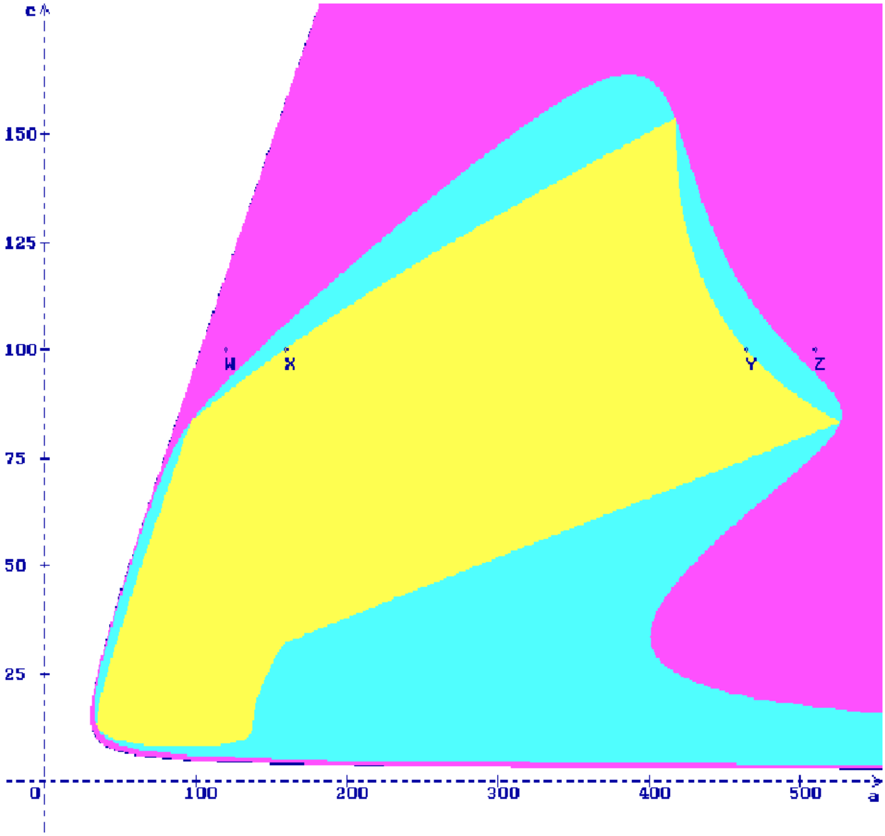


Fig. 10.34 Regions of sufficient stability condition and necessary and sufficient stability condition for (10.73): $p = 12$, $h = 0.024$, $\Delta = 0.012$

$p_{22} < 0$, $p_{11}p_{22} > p_{12}^2$, *then the trivial solution of the system* (10.93) *is asymptotically mean square stable.*

Let us show that by the conditions of Theorem 10.3 such numbers μ, γ, γ_1, γ_2 in fact exist. Choose γ from the condition $p_{12} = 0$, i.e.,

$$\gamma b_1 x_2^* - \mu a_1 x_1^* - a_2 x_1^* \left(1 - \Delta a_1 x_1^*\right) - \Delta\mu A b = 0. \tag{10.101}$$

Then via (10.86)

$$\gamma = \delta_{1\Delta}\mu + \delta_{2\Delta}, \tag{10.102}$$

where

$$\delta_{1\Delta} = \frac{a_2 b}{b_1}\left(\frac{a_1}{A b_1} + \Delta\right), \qquad \delta_{2\Delta} = \frac{a_2^2 b}{A b_1^2}\left(1 - \Delta\frac{a_1 b}{b_1}\right). \tag{10.103}$$

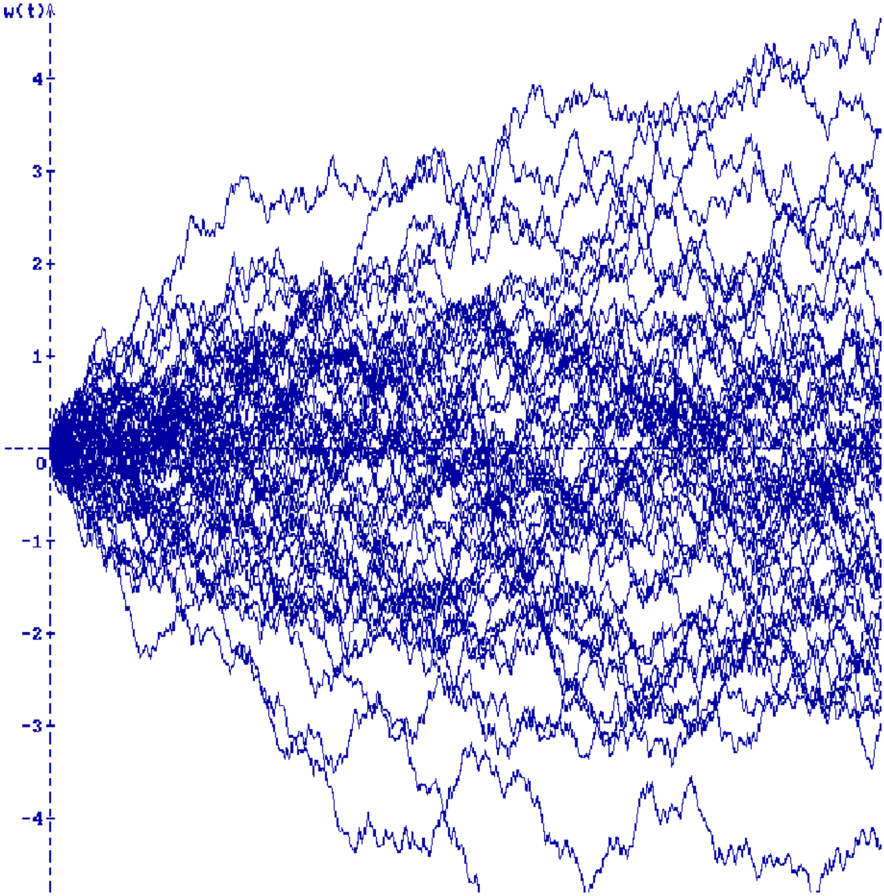


Fig. 10.35 Trajectories of the Wiener process

From the inequality $\gamma > \mu^2$, i.e., $\delta_{1\Delta}\mu + \delta_{2\Delta} > \mu^2$, it follows that the necessary number μ has to belong to the interval

$$\left(\frac{\delta_{1\Delta} - \sqrt{\delta_{1\Delta}^2 + 4\delta_{2\Delta}}}{2}, \frac{\delta_{1\Delta} + \sqrt{\delta_{1\Delta}^2 + 4\delta_{2\Delta}}}{2}\right). \tag{10.104}$$

Note that from (10.101) by the condition $\Delta a_1 x_1^* < 1$ or (which is the same due to (10.86))

$$\Delta < \frac{b_1}{a_1 b}, \tag{10.105}$$

it follows that

$$\gamma b_1 x_2^* - \mu a_1 x_1^* = a_2 x_1^* \left(1 - \Delta a_1 x_1^*\right) + \Delta \mu A b > 0. \tag{10.106}$$

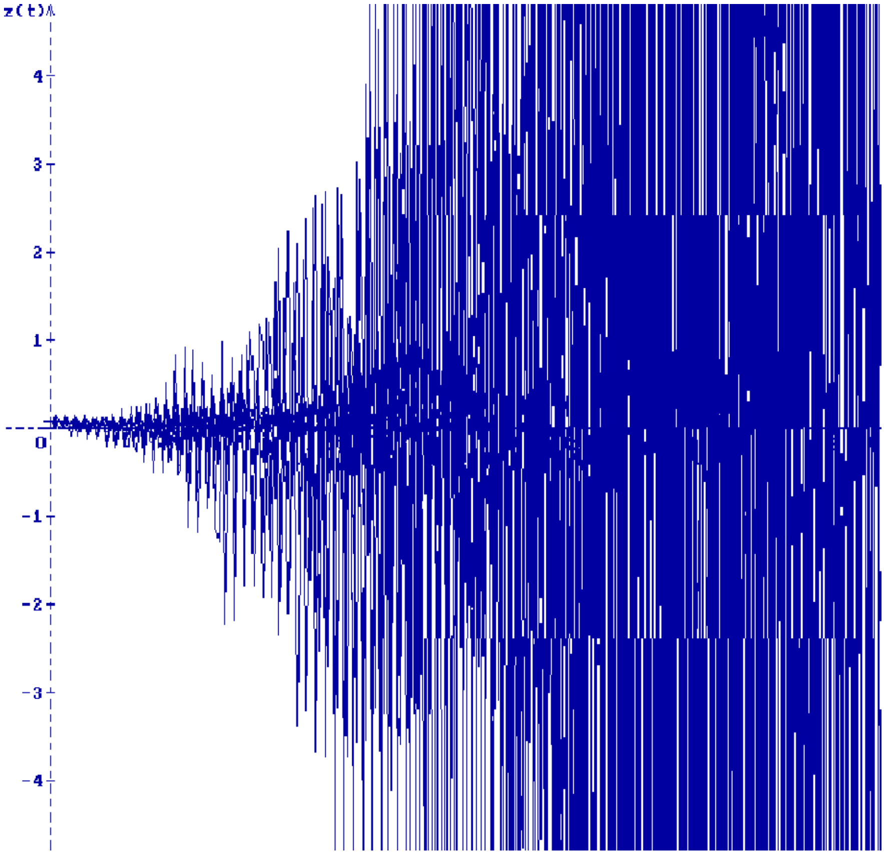


Fig. 10.36 Unstable solution of (10.73) in the point $W(120, 100)$, $a_0 = 0.1$

Supposing that $p_{11} < 0$ and using (10.102) and (10.106), we have

$$\begin{aligned} &- a_1x_1^*\left(2 - \Delta a_1x_1^*\right) + \sigma_1^2 + 2\mu b_1x_2^*\left(1 - \Delta a_1x_1^*\right) + (\delta_{1\Delta}\mu + \delta_{2\Delta})\Delta b_1^2\left(x_2^*\right)^2 \\ &\quad + \left[a_2b\left(1 - \Delta a_1x_1^*\right) + \Delta\mu Abb_1\right] \\ &\quad \times \left(2x_2^*h_1 + \gamma_1^{-1}x_1^*h_2\right) + \gamma_2^{-1}\mu Abh_1 < 0. \end{aligned} \tag{10.107}$$

From this via (10.105)

$$\mu < \frac{a_1x_1^*(2 - \Delta a_1x_1^*) - \sigma_1^2 - \Delta\delta_{2\Delta}b_1^2(x_2^*)^2 - b(1 - \Delta a_1x_1^*)(2Ah_1 + \gamma_1^{-1}a_2x_1^*h_2)}{2b_1x_2^*(1 - \Delta a_1x_1^*) + \Delta\delta_{1\Delta}b_1^2(x_2^*)^2 + \Delta Abb_1(2x_2^*h_1 + \gamma_1^{-1}x_1^*h_2) + \gamma_2^{-1}Abh_1}. \tag{10.108}$$

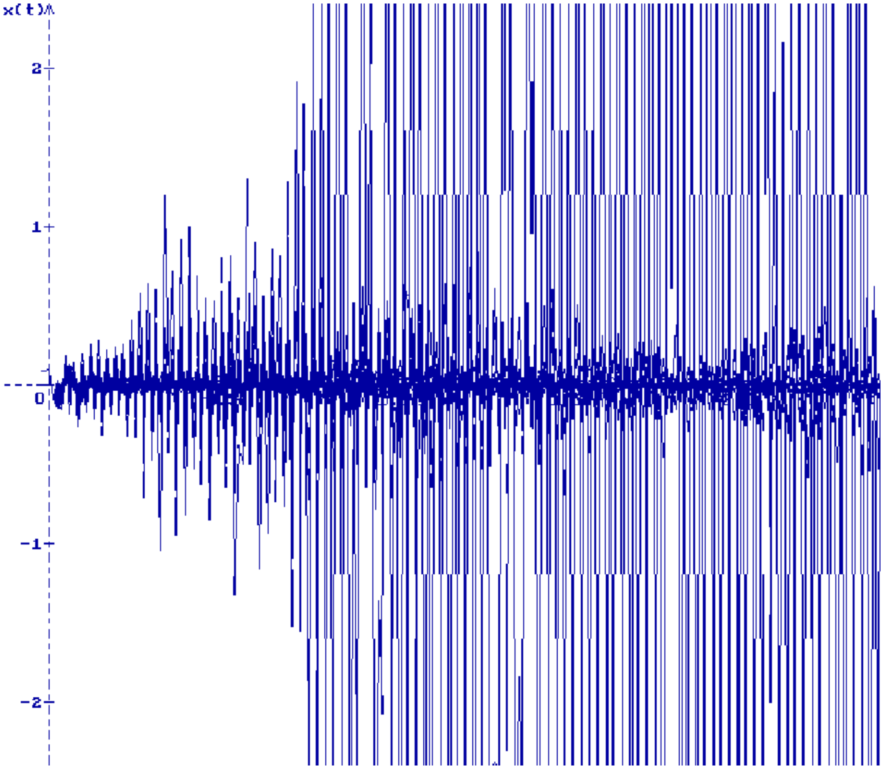


Fig. 10.37 Unstable solution of (10.73) in the point $Z(510, 100)$, $a_0 = 0.1$

Analogously, assuming that $p_{22} < 0$, we have

$$\Delta a_2^2 (x_1^*)^2 + \delta_2 \sigma_2^2 + \gamma_1 a_2 b x_1^* h_2 (1 - \Delta a_1 x_1^*) < \mu \left[2a_2 x_1^* (1 - \Delta a_1 x_1^*) - \delta_1 \sigma_2^2 - \gamma_1 \Delta A b^2 h_2 - \gamma_2 A b h_1 \right].$$

Considering the expression in the square brackets as a positive one, we obtain

$$\frac{\Delta a_2^2 (x_1^*)^2 + \delta_2 \sigma_2^2 + \gamma_1 a_2 b x_1^* h_2 (1 - \Delta a_1 x_1^*)}{2a_2 x_1^* (1 - \Delta a_1 x_1^*) - \delta_1 \sigma_2^2 - \gamma_1 \Delta A b^2 h_2 - \gamma_2 A b h_1} < \mu. \tag{10.109}$$

From (10.86) and (10.103) it follows that

$$\begin{aligned} \delta_{1\Delta} b_1^2 (x_2^*)^2 &= \frac{A^2 b_1^2}{a_2^2} \frac{b a_2}{b_1} \left(\frac{a_1}{A b_1} + \Delta \right) = \frac{Ab}{a_2} (a_1 + \Delta A b_1), \\ \delta_{2\Delta} b_1^2 (x_2^*)^2 &= \frac{A^2 b_1^2}{a_2^2} \frac{b a_2^2}{A b_1^2} (1 - \Delta a_1 x_1^*) = Ab \left(1 - \Delta \frac{a_1 b}{b_1} \right). \end{aligned} \tag{10.110}$$

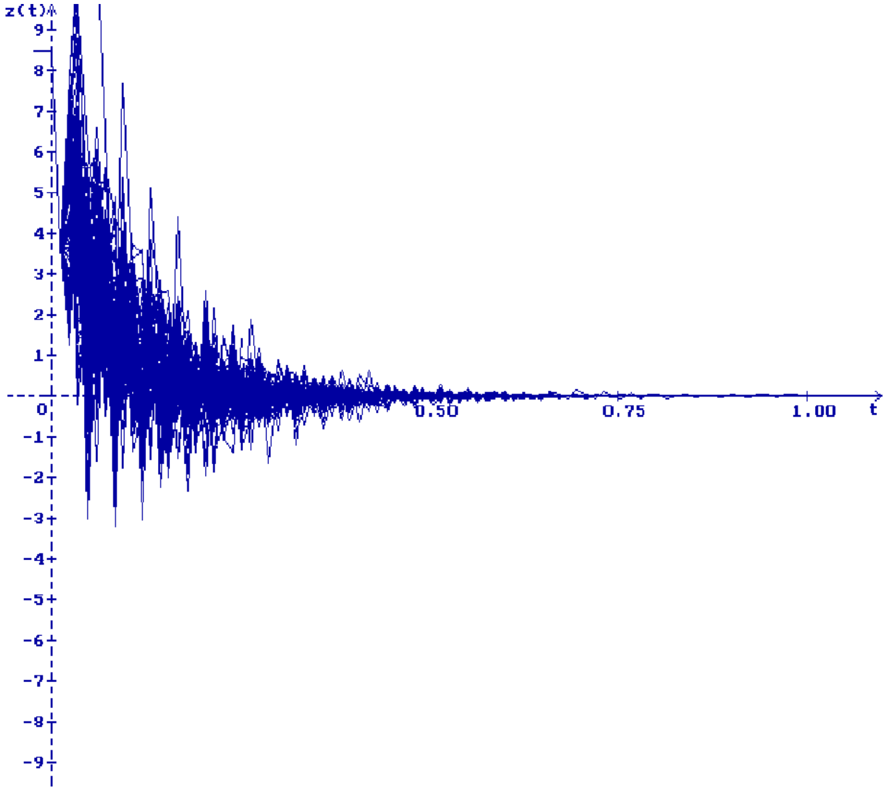


Fig. 10.38 Stable solution of (10.73) in the point $X(160, 100)$, $a_0 = 8.5$

Thus, from (10.108)–(10.110) it follows that the necessary μ has to belong to the interval

$$\left(\frac{A_{1\Delta} + B_{1\Delta}\gamma_1}{A_{4\Delta} - B_2\gamma_2 - B_{3\Delta}\gamma_1}, \frac{A_{2\Delta} - B_{1\Delta}\gamma_1^{-1}}{A_{3\Delta} + B_2\gamma_2^{-1} + B_{3\Delta}\gamma_1^{-1}}\right), \tag{10.111}$$

where via (10.86) and (10.110)

$$A_{1\Delta} = \frac{a_2^2 b\sigma_2^2}{Ab_1^2} + \Delta\frac{a_2^2 b^2}{b_1^2}\left(1 - \frac{a_1\sigma_2^2}{A}\right),$$

$$A_{2\Delta} = 2b\left(\frac{a_1}{b_1} - Ah_1\right) - \Delta b\left(\frac{a_1^2 b}{b_1^2} + A\left(1 - \Delta\frac{a_1 b}{b_1}\right)\right) - \sigma_1^2,$$

$$A_{3\Delta} = \frac{2Ab_1}{a_2} + \Delta\frac{Ab}{a_2}\big(Ab_1(2h_1 + \Delta) - a_1\big),$$

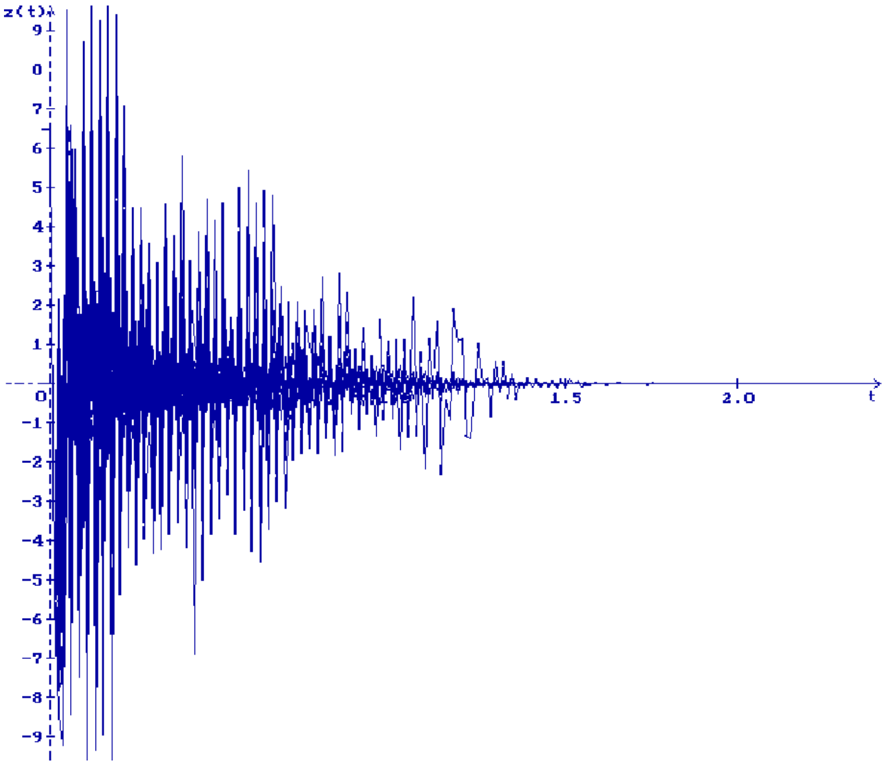


Fig. 10.39 Stable solution of (10.73) in the point $Y(465, 100)$, $a_0 = 6.5$

$$A_{4\Delta} = \frac{a_2 b}{b_1}\left(2(1 - bh_2) - \sigma_2^2\left(\frac{a_1}{Ab_1} + \Delta\right)\right),$$

$$B_{1\Delta} = \frac{a_2 b^2 h_2}{b_1}\left(1 - \Delta\frac{a_1 b}{b_1}\right), \qquad B_2 = Abh_1, \qquad B_{3\Delta} = \Delta Ab^2 h_2.$$

For the existence of the interval (10.111) it is necessary that the inequality

$$\frac{A_{1\Delta} + B_{1\Delta}\gamma_1}{A_{4\Delta} - B_2\gamma_2 - B_{3\Delta}\gamma_1} < \frac{A_{2\Delta} - B_{1\Delta}\gamma_1^{-1}}{A_{3\Delta} + B_2\gamma_2^{-1} + B_{3\Delta}\gamma_1^{-1}} \tag{10.112}$$

was correct or (which is the same)

$$\frac{A_{1\Delta} + B_{1\Delta}\gamma_1}{A_{2\Delta} - B_{1\Delta}\gamma_1^{-1}} \times \frac{A_{3\Delta} + B_2\gamma_2^{-1} + B_{3\Delta}\gamma_1^{-1}}{A_{4\Delta} - B_2\gamma_2 - B_{3\Delta}\gamma_1} < 1. \tag{10.113}$$

Optimal γ_1 and γ_2 are such that they give a minimum for the left part of the inequality (10.113). For simplification of the task of minimization, let us find a minimum of the left part of (10.113) by $\Delta = 0$. From (10.96) it follows that $A_{i0} = A_i$, $i = 1, \dots, 4$, $B_{10} = B_1$, $B_{30} = 0$. Thus, by $\Delta = 0$ the inequality (10.113) takes the form

$$\frac{A_1 + B_1\gamma_1}{A_2 - B_1\gamma_1^{-1}} \times \frac{A_3 + B_2\gamma_2^{-1}}{A_4 - B_2\gamma_2} < 1. \tag{10.114}$$

It is easy to show that the minimum of the left part of (10.114) equals $(\gamma_1/\gamma_2)^2$ by

$$\gamma_1 = \frac{\sqrt{B_1^2 + A_1A_2} + B_1}{A_2}, \qquad \gamma_2 = \frac{A_4}{\sqrt{B_2^2 + A_3A_4} + B_2}. \tag{10.115}$$

Substituting (10.115) into (10.114), we obtain (10.91). So, by $\Delta = 0$, the inequalities (10.113) and (10.112) with γ_1 and γ_2, which are defined in (10.115), hold. Because of the continuous dependence of the left part of inequality (10.113) on Δ and for small enough $\Delta > 0$, we have

$$\frac{A_{1\Delta} + B_{1\Delta}\gamma_{1\Delta}}{A_{4\Delta} - B_2\gamma_{2\Delta} - B_{3\Delta}\gamma_{1\Delta}} < \frac{A_{2\Delta} - B_{1\Delta}\gamma_{1\Delta}^{-1}}{A_{3\Delta} + B_2\gamma_{2\Delta}^{-1} + B_{3\Delta}\gamma_{1\Delta}^{-1}}, \tag{10.116}$$

where

$$\gamma_{1\Delta} = \frac{\sqrt{B_{1\Delta}^2 + A_{1\Delta}A_{2\Delta}} + B_{1\Delta}}{A_{2\Delta}}, \qquad \gamma_{2\Delta} = \frac{A_{4\Delta}}{\sqrt{B_2^2 + A_{3\Delta}A_{4\Delta}} + B_2}. \tag{10.117}$$

Thus, the interval (10.111) by γ_1, γ_2, which are defined in (10.117), and a small enough $\Delta > 0$ exist.

Let us show that the intervals (10.111) and (10.117) belong to the interval (10.104). Since the left bound of interval (10.104) by the condition (10.105) is negative and the left bound of interval (10.111) and (10.117) is positive, it is enough to show that

$$\frac{A_{2\Delta} - B_{1\Delta}\gamma_{1\Delta}^{-1}}{A_{3\Delta} + B_2\gamma_{2\Delta}^{-1} + B_{3\Delta}\gamma_{1\Delta}^{-1}} < \frac{\delta_{1\Delta} + \sqrt{\delta_{1\Delta}^2 + 4\delta_{2\Delta}}}{2}. \tag{10.118}$$

On the other hand to prove (10.118) it is enough to show that, for any Δ that satisfies (10.105), the inequality

$$\frac{A_{2\Delta} - B_{1\Delta}\gamma_{1\Delta}^{-1}}{A_{3\Delta} + B_2\gamma_{2\Delta}^{-1} + B_{3\Delta}\gamma_{1\Delta}^{-1}} < \delta_{1\Delta}.$$

holds. In fact, via (10.103) and (10.105)

$$\begin{aligned}
&\frac{A_{2\Delta} - B_{1\Delta}\gamma_{1\Delta}^{-1}}{A_{3\Delta} + B_2\gamma_{2\Delta}^{-1} + B_{3\Delta}\gamma_{1\Delta}^{-1}} \\
&\quad \le \frac{A_{2\Delta}}{A_{3\Delta}} \le \frac{\frac{2ba_1}{b_1} - \frac{\Delta b^2 a_1^2}{b_1^2}}{\frac{2Ab_1}{a_2} + \frac{\Delta Ab}{a_2}(\Delta Ab_1 - a_1)} \\
&\quad = \frac{a_1 a_2 b(2b_1 - \Delta b a_1)}{Ab_1^2(2b_1 + \Delta b(\Delta Ab_1 - a_1))} \\
&\quad = \frac{a_1 a_2 b}{Ab_1^2} \times \frac{2b_1 - \Delta b a_1}{2b_1 - \Delta b a_1 + \Delta^2 Abb_1} < \frac{a_1 a_2 b}{Ab_1^2} < \delta_{1\Delta}.
\end{aligned}$$

So, the following theorem is proven.

Theorem 10.4 *If conditions (10.85) and (10.91) hold and the step of discretization Δ satisfies the conditions (10.105) and (10.116) then the trivial solution of the system (10.93) is asymptotically mean square stable and, respectively, the positive point of equilibrium (10.86) of the system (10.92) is stable in probability.*

Example 10.3 The solution of the system (10.87) was numerically simulated by virtue of the difference analogue (10.92) by the following values of the parameters: $a = 5$, $a_1 = 0.2$, $a_2 = 0.5$, $b_1 = 0.5$, $b = 3.5$, $h_1 = h_2 = 0.02$, $\sigma_1 = 0.6$, $\sigma_2 = 0.8$, $\varphi_1(s) = 8.5\cos s$, $\varphi_2(s) = 8.5\sin(s+1)$. Here $A = 3.6$, $A_1 = 0.622$, $A_2 = 1.936$, $A_3 = 7.2$, $A_4 = 6.261$, $B_1 = 0.245$, $B_2 = 0.252$, $x_1^* = 7$, $x_2^* = 7.2$, $\varphi_1(-0.02) = 8.498$, $\varphi_1(-0.01) = 8.499$, $\varphi_1(0) = 8.5$, $\varphi_2(-0.02) = 7.059$, $\varphi_2(-0.01) = 7.106$ and $\varphi_2(0) = 7.152$. The conditions (10.85) and (10.91) hold, and the condition (10.105) takes the form $\Delta < 0.714$. By choosing the step of discretization $\Delta = 0.01$, the condition (10.116) holds too: $0.152 < 0.253$. One thousand trajectories of the system (10.87) were simulated and all of them had convergence to the equilibrium point (x_1^*, x_2^*). In Fig. 10.40 one of the trajectories obtained is shown in the phase space (x_1, x_2), and in Fig. 10.41 the dependence of $x_1(t)$ and $x_2(t)$ on time is shown.

10.5 Difference Analogues of an Integro-Differential Equation of Convolution Type

Here the reliability to preserve the property of stability is considered for some difference analogues of the deterministic nonlinear integro-differential equation of convolution type. Difference analogues are constructed both with discrete and with continuous time.

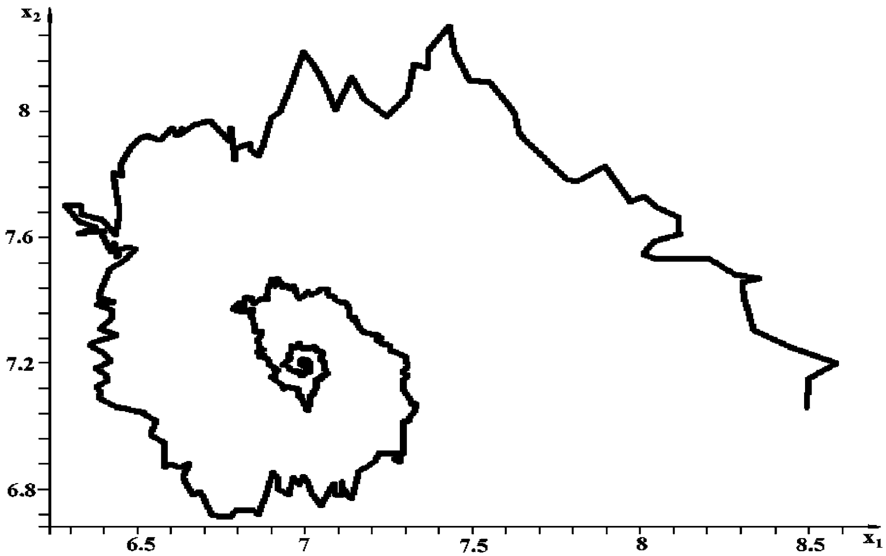


Fig. 10.40 Trajectory of solution of the system (10.87) in the phase space (x_1, x_2)

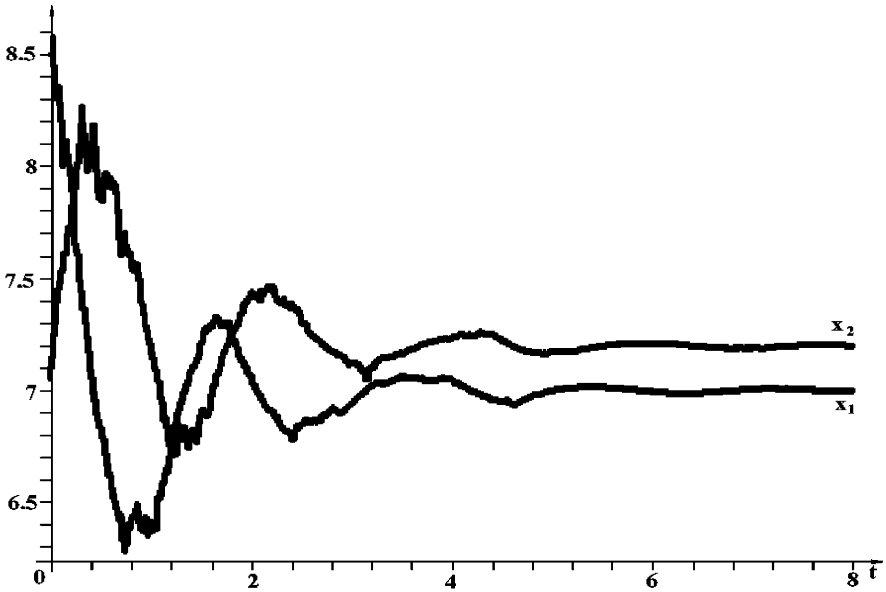


Fig. 10.41 Dependence of $x_1(t)$ and $x_2(t)$ on time

The nonlinear integro-differential equations of convolution type

$$\dot{x}(t) = \int_0^t K(t-s) f\big(x(s)\big)\,\mathrm{d}s$$

arise usually in some problems related to evolutionary processes in ecology, in nuclear reactors, in control theory etc. [60, 61, 66–68, 71, 72, 78–81, 90, 91, 176]. Below the equation of convolution type with exponential kernel is considered,

$$\dot{x}(t) = -k\int_0^t e^{-\lambda(t-s)} f\big(x(s)\big)\,ds \tag{10.119}$$

with $k > 0$, $\lambda > 0$ and

$$f(x) = \sum_{i=1}^{m} \alpha_i x^{\nu_i}, \quad \nu_i = \frac{2p_i+1}{2q_i+1},\ p_i \ge q_i \ge 0, \tag{10.120}$$

where $\alpha_i > 0$, p_i and q_i are integers.

It is easy to check that the trivial solution of (10.119) is stable. Indeed, putting $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, one can transform (10.119) into the system of equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -kf\big(x_1(t)\big) - \lambda x_2(t).\end{aligned}$$

The function

$$V(t) = k\sum_{i=1}^{m} \frac{\alpha_i}{\mu_i} x_1^{2\mu_i}(t) + x_2^2(t), \quad \mu_i = \frac{1}{2}(\nu_i+1) = \frac{p_i+q_i+1}{2q_i+1} \ge 1, \tag{10.121}$$

is a Lyapunov function for this system since $V(t) > 0$ for $x_1^2(t) + x_2^2(t) > 0$, and $\dot{V}(t) = -2\lambda x_2^2(t) < 0$ unless $x_2(t) = 0$.

10.5.1 Some Difference Analogues with Discrete Time

It is obvious that one differential equation can have several difference analogues, according to the choice of a numerical scheme. However, not all of these analogues need to be asymptotically stable. The problem is to determine which methods can be employed with the expectation that the difference analogue will preserve the qualitative behavior of the solutions of the original problem. In particular, the question is how one may construct a difference analogue of a continuous asymptotically stable system that will be asymptotically stable too.

In this section three possible schemes are proposed for the construction of difference analogues of the integro-differential equation

$$\dot{x}(t) = -k\int_0^t e^{-\lambda(t-s)} x^3(s)\,ds. \tag{10.122}$$

This is a particular case of (10.119) with $m = 1$, $\alpha_1 = 1$, $p_1 = 1$, $q_1 = 0$. Conditions for the asymptotic stability of the trivial solutions of these difference analogues are obtained.

Scheme 10.1 Divide the interval $[0,t]$ into $i+1$ intervals of length $\Delta > 0$. In this way $t=(i+1)\Delta$, $s=j\Delta$, $j=0,1,\ldots,i,i+1$, $x_j=x(j\Delta)$. Using the left-hand difference derivative $(x_{i+1}-x_i)/\Delta$ for the representation of $\dot{x}(t)$ in the point $t=(i+1)\Delta$, as a result we obtain the difference analogue of (10.122) in the form

$$x_{i+1}=x_i-k\Delta^2\sum_{j=0}^{i}\mathrm{e}^{-\lambda\Delta(i+1-j)}x_j^3,\quad i=0,1,\ldots. \tag{10.123}$$

Denoting $a=\mathrm{e}^{-\lambda\Delta}$, we transform the right-hand side of (10.123) in the following way:

$$x_{i+1}=x_i-ak\Delta^2x_i^3+a\left(-k\Delta^2\sum_{j=0}^{i-1}\mathrm{e}^{-\lambda\Delta(i-j)}x_j^3\right).$$

Applying (10.123) for $i-1$ leads to

$$\begin{aligned}&x_1=x_0-ak\Delta^2x_0^3,\\&x_{i+1}=x_i-ak\Delta^2x_i^3+a(x_i-x_{i-1}),\quad i=1,2,\ldots.\end{aligned} \tag{10.124}$$

Equation (10.124) has four parameters: k, λ, Δ, x_0. Putting

$$y_i=\frac{x_i}{x_0},\qquad \tau=\lambda\Delta,\qquad \gamma=k\frac{x_0^2}{\lambda^2},\qquad a=\mathrm{e}^{-\tau},\qquad b=\gamma\tau^2, \tag{10.125}$$

we finally obtain the equation with two parameters only:

$$\begin{aligned}&y_0=1,\qquad y_1=1-ab,\\&y_{i+1}=y_i-aby_i^3+a(y_i-y_{i-1}),\quad i=1,2,\ldots.\end{aligned} \tag{10.126}$$

Scheme 10.2 Divide the interval $[0,t]$ into i intervals of length $\Delta > 0$. In this way $t=i\Delta$, $s=j\Delta$, $j=0,1,\ldots,i$, $x_j=x(j\Delta)$. Using the right-hand difference derivative $(x_{i+1}-x_i)/\Delta$ for the representation of $\dot{x}(t)$ in the point $t=i\Delta$, as a result, similarly to (10.124), we obtain the difference analogue of (10.122) in the form

$$x_{i+1}=x_i-k\Delta^2\sum_{j=1}^{i}\mathrm{e}^{-\lambda\Delta(i-j)}x_j^3,\quad i=0,1,\ldots,$$

or

$$\begin{aligned}&x_1=x_0,\\&x_{i+1}=x_i-k\Delta^2x_i^3+\mathrm{e}^{-\lambda\Delta}(x_i-x_{i-1}),\quad i=1,2,\ldots.\end{aligned} \tag{10.127}$$

Following the same approach as above and using (10.125), one can represent this difference equation in the two-parameter form

$$\begin{aligned} &y_1 = y_0 = 1, \\ &y_{i+1} = y_i - by_i^3 + a(y_i - y_{i-1}), \quad i = 1, 2, \dots. \end{aligned} \tag{10.128}$$

Scheme 10.3 Divide the interval $[0, t]$ into $i + 1$ intervals of length $\Delta > 0$. In this way $t = (i + 1)\Delta$, $s = j\Delta$, $j = 0, 1, \dots, i, i + 1$, $x_j = x(j\Delta)$. Using the left-hand difference derivative $(x_{i+1} - x_i)/\Delta$ for the representation of $\dot{x}(t)$ in the point $t = (i + 1)\Delta$, as a result, analogously to (10.127), we obtain the difference equation in the form

$$x_{i+1} = x_i - k\Delta \sum_{j=1}^{i+1} e^{-\lambda\Delta(i+1-j)} x_j^3, \quad i = 0, 1, \dots,$$

or

$$\begin{aligned} &x_1 = x_0 - k\Delta^2 x_1^3, \\ &x_{i+1} = x_i - k\Delta^2 x_{i+1}^3 + e^{-\lambda\Delta}(x_i - x_{i-1}), \quad i = 1, 2, \dots. \end{aligned} \tag{10.129}$$

Using (10.125) in the two-parameter form we obtain

$$\begin{aligned} &y_0 = 1, \qquad y_1 = 1 - by_1^3, \\ &y_{i+1} = y_i - by_{i+1}^3 + a(y_i - y_{i-1}), \quad i = 1, 2, \dots. \end{aligned} \tag{10.130}$$

Remark 10.7 If $b = 0$, then (10.126), (10.128) and (10.130) coincide with the equation $y_{i+1} = (1 + a)y_i - ay_{i-1}$, which has a stable but not an asymptotically stable trivial solution.

10.5.2 The Construction of the Lyapunov Functionals

We describe here in detail the construction of Lyapunov functionals V_i with the condition $\Delta V_i < 0$ for (10.126), (10.128) and (10.130). Via the procedure of the construction of the Lyapunov functionals we have the following four steps.

Step 1 Using the procedure of the construction of the Lyapunov functionals represent (10.126) in the form

$$y_{i+1} = F_{1i} + \Delta F_{3i}, \quad i = 0, 1, \dots,$$

where

$$F_{1i} = y_i - aby_i^3, \quad i = 0, 1, \dots,$$

$$F_{30} = 0, \qquad \Delta F_{30} = ay_0,$$
$$F_{3i} = ay_{i-1}, \qquad \Delta F_{3i} = a(y_i - y_{i-1}), \quad i = 1, 2, \dots.$$

Step 2 Consider the auxiliary difference equation without delay

$$z_0 = 1, \qquad z_{i+1} = z_i - abz_i^3, \quad i = 0, 1, \dots. \tag{10.131}$$

The function $v(z) = z^2$ is a Lyapunov function for this equation. In fact, using (10.131) we get

$$\begin{aligned}\Delta v_i &= v(z_{i+1}) - v(z_i) = z_{i+1}^2 - z_i^2 \\ &= \left(z_i - abz_i^3\right)^2 - z_i^2 = (ab)^2 z_i^4\left(z_i^2 - \frac{2}{ab}\right).\end{aligned}$$

Since $z_0 = 1$, via the condition $ab < 2$ we have $\Delta v_i < 0$ for all $i = 0, 1, \dots$ provided $z_i \neq 0$.

Step 3 We will construct the Lyapunov functional V_i for the system (10.126) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{10} = v(y_0) = y_0^2, \qquad V_{1i} = v(y_i - F_{3i}) = (y_i - ay_{i-1})^2, \quad i = 1, 2, \dots$$

Calculating ΔV_{1i}, $i = 1, 2, \dots$, via (10.126) and Lemma 1.1, we obtain

$$\begin{aligned}\Delta V_{1i} &= (y_{i+1} - ay_i)^2 - (y_i - ay_{i-1})^2 \\ &= \left(y_i - aby_i^3 - ay_{i-1}\right)^2 - (y_i - ay_{i-1})^2 \\ &= (ab)^2 y_i^6 - 2aby_i^4 + 2a^2by_i^3 y_{i-1} \\ &\le (ab)^2 y_i^6 - a^2b\left(\frac{2}{a} - \frac{3}{2}\right)y_i^4 + \frac{1}{2}a^2by_{i-1}^4.\end{aligned}$$

Analogously for $i = 0$:

$$\begin{aligned}\Delta V_{10} &= y_1^2 - y_0^2 = (1 - ab)^2 - 1 \\ &= -(ab)^2\left(\frac{2}{ab} - 1\right) < -(ab)^2\left(g_1(\tau) - 1\right),\end{aligned}$$

where, via (10.125),

$$g_1(\tau) = \frac{2(e^\tau - 1)}{\gamma\tau^2}. \tag{10.132}$$

Step 4 Choosing the additional functional V_{2i} in the form $V_{20} = 0$, $V_{2i} = \frac{1}{2}a^2by_{i-1}^4$, $i = 1, 2, \dots$, for the functional $V_i = V_{1i} + V_{2i}$ we get

$$\Delta V_i \le -(ab)^2 y_i^4\left(g_1(\tau) - y_i^2\right), \quad i = 0, 1, \dots. \tag{10.133}$$

It is easy to see that

$$\lim_{\tau\to 0} g_1(\tau)=\infty, \qquad \lim_{\tau\to\infty} g_1(\tau)=\infty,$$

$$\inf_{\tau\ge 0} g_1(\tau)=g_1(\tau_0)=\frac{2}{(2-\tau_0)\tau_0\gamma}\approx\frac{3.088}{\gamma},$$

where $\tau_0\approx 1.594$ is the root of the equation

$$2=(2-\tau)\,e^{\tau}. \tag{10.134}$$

Let us suppose that the sequence y_i^2 is bounded and there exists a $\tau>0$ such that

$$y_i^2<g_1(\tau), \quad i=0,1,\dots. \tag{10.135}$$

In this way, $\Delta V_i<0$ for all $i=0,1,\dots$, while $y_i\neq 0$. If the sequence y_i^2 is bounded by $g_1(\tau_0)$, where τ_0 is the root of (10.134), then (10.135) is correct for all $\tau>0$. If y_i^2 is bounded by some $M>g_1(\tau_0)$, then (10.135) is correct for $\tau\in(0,\tau_1)\cup(\tau_2,\infty)$, where τ_1 and τ_2 are two positive roots of the equation

$$2\bigl(e^{\tau}-1\bigr)=M\gamma\tau^2. \tag{10.136}$$

The corresponding analysis for (10.128) proceeds as follows.

Step 1. We choose $F_{1i}=y_i-by_i^3$ and $F_{3i}=ay_{i-1}$.
Step 2. This step is the same as before by the condition $b<2$.
Step 3. One can show that

$$\Delta V_{10}=(y_1-ay_0)^2-y_0^2=a^2-2a=a(a-2)<0,$$

$$\Delta V_{1i}\le b^2y_i^6-by_i^4\left(2-\frac{3a}{2}\right)+\frac{1}{2}aby_{i-1}^4, \quad i=1,2,\dots.$$

Step 4. Put

$$V_{20}=0, \qquad V_{2i}=\frac{1}{2}aby_{i-1}^4, \quad i=1,2,\dots.$$

Then

$$\Delta V_i\le -b^2y_i^4\bigl(g_2(\tau)-y_i^2\bigr),$$

where

$$g_2(\tau)=\frac{2(1-e^{-\tau})}{\gamma\tau^2}. \tag{10.137}$$

The function $g_2(\tau)$ is a strictly decreasing one for $\tau>0$ and

$$\lim_{\tau\to 0} g_2(\tau)=\infty, \qquad \lim_{\tau\to\infty} g_2(\tau)=0.$$

Thus, if the sequence y_i^2 is bounded by some $M > 0$ then the condition $y_i^2 < g_2(\tau)$, $i = 0, 1, \dots$ (and therefore $\Delta V_i < 0$) is correct for $\tau \in (0, \tau_0)$, where τ_0 is the positive root of the equation

$$2\big(1 - \mathrm{e}^{-\tau}\big) = M\gamma\tau^2. \tag{10.138}$$

Finally, for (10.130) we have the following.

Step 1. Choose $F_{1i} = y_i - by_i^3$ and $F_{30} = -by_0^3 = -b$, $F_{3i} = ay_{i-1} - by_i^3$, $\Delta F_{3i} = a(y_i - y_{i-1}) - b(y_{i+1}^3 - y_i^3)$, $i = 1, 2, \dots$.

Step 2. This step is the same as before by the condition $b < 2$.

Step 3. One can show that

$$\begin{aligned}\Delta V_{10} &= (y_1 - F_{31})^2 - (y_0 - F_{30})^2 = (1-a)^2 - (1+b)^2\\ &= -a(2-a) - b(2+b) < 0\end{aligned}$$

and

$$\begin{aligned}\Delta V_{1i} &= (y_{i+1} - F_{3,i+1})^2 - (y_i - F_{3i})^2\\ &= \big(y_{i+1} - ay_i + by_{i+1}^3\big)^2 - \big(y_i - ay_{i-1} + by_i^3\big)^2\\ &= (y_i - ay_{i-1})^2 - \big(y_i - ay_{i-1} + by_i^3\big)^2\\ &= -2by_i^4 - b^2y_i^6 + 2aby_i^3y_{i-1} \le -2by_i^4 + 2aby_i^3y_{i-1}\\ &\le -by_i^4\left(2 - \frac{3a}{2}\right) + \frac{1}{2}aby_{i-1}^4, \quad i = 1, 2, \dots.\end{aligned}$$

Step 4. Put

$$V_{20} = 0, \qquad V_{2i} = \frac{1}{2}aby_{i-1}^4, \quad i = 1, 2, \dots.$$

Thus, for all $\tau > 0$ we obtain $\Delta V_i \le -2b(1-a)y_i^4 < 0$.

10.5.3 Proof of Asymptotic Stability

Here we can show how the functional V_i constructed above can be used to give the desired conclusion. We give here the analysis for the cases (10.126) and (10.124).

From (10.133) and (10.135) it follows that

$$\sum_{j=0}^{i} \Delta V_j = V_{i+1} - V_0 \le -(ab)^2 \sum_{j=0}^{i} y_j^4\big(g_1(\tau) - y_j^2\big) < 0. \tag{10.139}$$

Therefore, $0 \le V_{i+1} \le V_0 = y_0^2 = 1$. Moreover, $V_{i+1} \ge V_{2i} = \frac{1}{2}a^2 b y_i^4$. From this, via (10.125),

$$x_i^2 \le \left(\frac{2}{k}\right)^{\frac{1}{2}} \frac{|x_0|}{a\Delta}.$$

So, for any $\varepsilon > 0$ there exists $\delta = (\frac{k}{2})^{\frac{1}{2}} a\Delta\varepsilon^2$ such that $|x_i| < \varepsilon$, $i > 0$, if $|x_0| < \delta$. In other words, we have shown that the trivial solution of (10.124) is stable.

Besides from (10.139) and $V_{i+1} \ge 0$ it follows that

$$\sum_{j=0}^{\infty} y_j^4 \big(g_1(\tau) - y_j^2\big) \le \frac{V_0}{(ab)^2}. \tag{10.140}$$

The convergence of the series in the left-hand part of (10.140) implies that

$$\lim_{i\to\infty} y_i^4 \big(g_1(\tau) - y_i^2\big) = 0.$$

It means that either $\lim_{i\to\infty} y_i^4 = 0$ or $\lim_{i\to\infty} y_i^2 = g_1(\tau)$. In any case the limit of y_i by $i \to \infty$ exists. From (10.126) it follows that $\lim_{i\to\infty} y_i = 0$. Via (10.125), the solution of (10.124) satisfies the condition $\lim_{i\to\infty} x_i = 0$. The proof is completed.

Remark 10.8 A similar argument applies to solutions of (10.128) and (10.130).

We summarize our conclusions in the following way. Assume that k, λ, x_0 are given; we investigate the solutions x_i of (10.124), (10.127) and (10.129) for a fixed value of $\Delta > 0$.

Theorem 10.5 *If the solution x_i of* (10.124) *satisfies the condition $x_i^2 \le g_1(\tau_0)x_0^2$, where τ_0 is the root of* (10.134), *then $x_i \to 0$, regardless of the step size $\Delta > 0$. If the solution x_i of* (10.124) *satisfies the condition $x_i^2 \le Mx_0^2$ for some $M > g_1(\tau_0)$, then $x_i \to 0$ for all $\Delta \in (0, \frac{\tau_1}{\lambda}) \cup (\frac{\tau_2}{\lambda}, \infty)$, where τ_1 and τ_2 are the roots of* (10.136). *If the solution x_i of* (10.127) *satisfies the condition $x_i^2 \le Mx_0^2$ for some $M > 0$, then $x_i \to 0$ for all $\Delta \in (0, \frac{\tau_0}{\lambda})$, where τ_0 is the root of* (10.138). *The solution x_i of* (10.129) *converges to zero for all $\Delta > 0$.*

Remark 10.9 In the statements of Theorem 10.5 we have considered the behavior of bounded solutions of the discrete equations. We can observe (see, e.g., [75, 76]) that unbounded solutions may arise with particular combinations of x_0, Δ, λ. Our calculations indicate that if $g_1(\tau) > 1$, then the solution of (10.126) satisfies the condition $|y_i| < 1$, $i = 1, 2, \dots$. In Figs. 10.42–10.44 one can see the behavior of the solution of (10.126) with the different values of the parameters τ, γ and the function $g_1(\tau)$: we have in Fig. 10.42 ($\tau = 0.1$, $\gamma = 6$, $g_1(\tau) = 3.51$), Fig. 10.43 ($\tau = 0.01$, $\gamma = 15$, $g_1(\tau) = 13.4$) and Fig. 10.44 ($\tau = 0.1$, $\gamma = 20$, $g_1(\tau) = 1.05$). In Fig. 10.45 it is shown that by $\tau = 0.24$, $\gamma = 25$ the solutions of (10.126) (number 1,

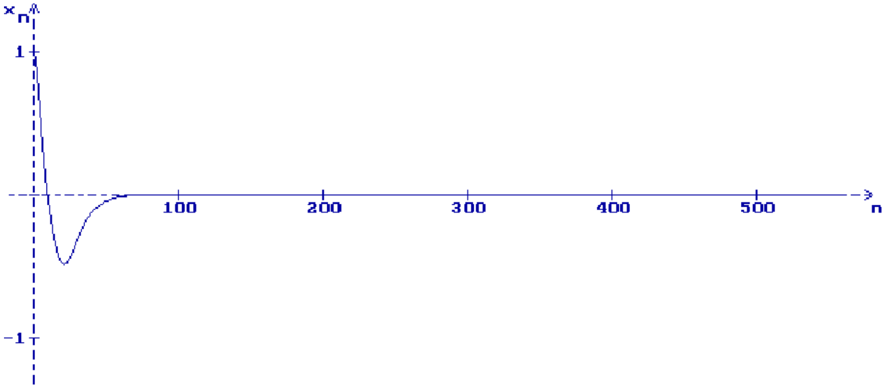


Fig. 10.42 The solution of (10.126): $\tau = 0.1$, $\gamma = 6$, $g_1(\tau) = 3.51$

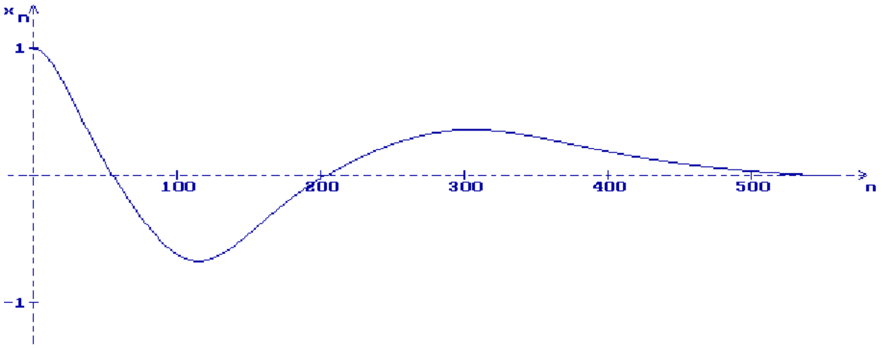


Fig. 10.43 The solution of (10.126): $\tau = 0.01$, $\gamma = 15$, $g_1(\tau) = 13.4$

$g_1(\tau) = 0.38$) and (10.130) (number 3) converge to zero, but the solution of (10.128) (number 2, $g_2(\tau) = 0.3$) goes to infinity.

Remark 10.10 Note that the difference schemes considered above can be constructed and for more general nonlinear integro-differential equation we have

$$\dot{x}(t) = -k \int_0^t e^{-\lambda(t-s)} x^r(s)\,ds, \quad k, \lambda > 0, \tag{10.141}$$

where r is an arbitrary odd number. The equations of the type of difference analogue (10.124) and (10.126), respectively, for (10.141) are

$$x_1 = x_0 - akh^2 x_0^r,$$
$$x_{i+1} = x_i - ak\Delta^2 x_i^r + a(x_i - x_{i-1}), \quad i = 1, 2, \ldots$$

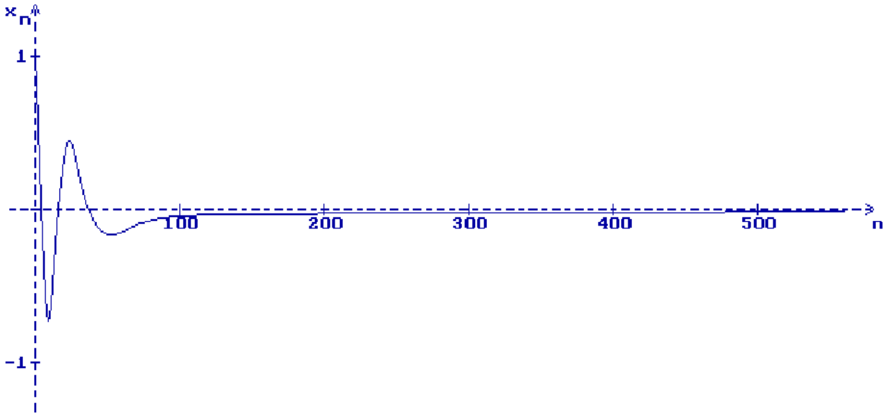


Fig. 10.44 The solution of (10.126): $\tau = 0.1$, $\gamma = 20$, $g_1(\tau) = 1.05$

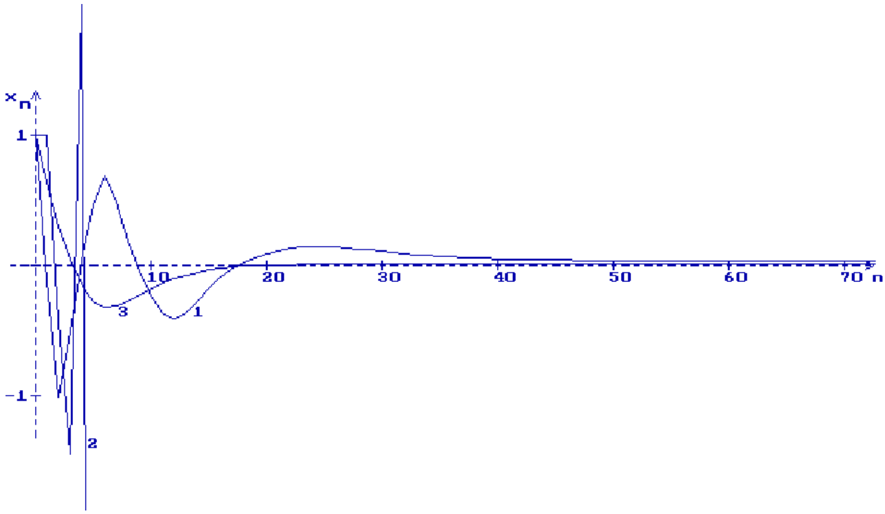


Fig. 10.45 The solutions of (10.126), (10.128) and (10.130): $\tau = 0.24$, $\gamma = 25$, $g_1(\tau) = 0.38$, $g_2(\tau) = 0.3$

and

$$y_0 = 1, \qquad y_1 = 1 - \gamma a \Delta^2,$$
$$y_{i+1} = y_i - abx_i^r + a(y_i - y_{i-1}), \quad i = 1, 2, \ldots,$$

where $\gamma = k\frac{y_0^{r-1}}{\lambda^2}$ and y_i, τ, a, b are defined by (10.125). By that the functional

$$V_i = (y_i - ay_{i-1})^2 + \frac{2a^2 b}{r+1} y_{i-1}^{r+1}, \quad i = 1, 2, \ldots,$$

satisfied the condition $\Delta V_i \leq -(ab)^2 y_i^{r+1}(g_1(\tau) - y_i^{r-1})$ with $g_1(\tau)$ defined by (10.132).

10.5.4 Difference Analogue with Continuous Time

To construct the difference analogue of (10.119) with continuous time rewrite this equation in the equivalent form

$$\dot{x}(t) = -k \int_0^t e^{-\lambda s} f\big(x(t-s)\big)\, ds. \tag{10.142}$$

Let Δ be a small enough positive number. Using the representation (10.119) for $t \in [0, \Delta)$ and (10.142) for $t \geq \Delta$, we can construct a difference analogue in the form of the following difference equation with continuous time:

$$\begin{aligned} x(t) &= x(0) - kt^2 e^{-\lambda t} f\big(x(0)\big), \quad t \in [0, \Delta), \\ x(t+\Delta) &= x(t) - k\Delta^2 F(t), \quad t \geq 0, \\ F(t) &= \sum_{j=0}^{[\frac{t}{\Delta}]} e^{-\lambda \Delta j} f\big(x(t - j\Delta)\big). \end{aligned} \tag{10.143}$$

If $t \in [0, \Delta)$ then $F(t) = f(x(t))$. For $t \geq \Delta$ transform $F(t)$ in the following way:

$$\begin{aligned} F(t) &= f\big(x(t)\big) + \sum_{j=1}^{[\frac{t}{\Delta}]} e^{-\lambda \Delta j} f\big(x(t - j\Delta)\big) \\ &= f\big(x(t)\big) + \sum_{j=0}^{[\frac{t}{\Delta}]-1} e^{-\lambda \Delta (j+1)} f\big(x\big(t - (j+1)\Delta\big)\big) \\ &= f\big(x(t)\big) + e^{-\lambda\Delta} \sum_{j=0}^{[\frac{t-\Delta}{\Delta}]} e^{-\lambda \Delta j} f\big(x(t - \Delta - j\Delta)\big) \\ &= f\big(x(t)\big) + e^{-\lambda\Delta} F(t - \Delta). \end{aligned} \tag{10.144}$$

It follows from (10.143) that

$$F(t) = -\frac{x(t+\Delta) - x(t)}{k\Delta^2}, \qquad F(t-\Delta) = -\frac{x(t) - x(t-\Delta)}{k\Delta^2}.$$

Substituting $F(t)$ and $F(t-\Delta)$ from this into (10.144), we transform (10.143) into the form

$$x(t+\Delta) = x(t) - k\Delta^2 f\big(x(t)\big) + e^{-\lambda\Delta}\big(x(t) - x(t-\Delta)\big), \quad t > \Delta. \tag{10.145}$$

The process $x(t)$ is defined by (10.145) for $t > t_0 = 2\Delta$ with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in [t_0 - 2\Delta, t_0] = [0, 2\Delta], \tag{10.146}$$

where

$$\phi(\theta) = \begin{cases} x(0) - k\theta^2 e^{-\lambda\theta} f(x(0)), & \theta \in [t_0 - 2\Delta, t_0 - \Delta) = [0, \Delta), \\ \phi(\theta - \Delta) - k\Delta^2 f(\phi(\theta - \Delta)), & \theta \in [t_0 - \Delta, t_0] = [\Delta, 2\Delta]. \end{cases}$$

Note that via (10.120) the order of nonlinearity of (10.145) is, generally speaking, higher than 1.

Definition 10.1 The solution of (10.145) with initial condition (10.146) is called asymptotically quasitrivial if $\lim_{j\to\infty} x(t + j\Delta) = 0$ for each $t \in [t_0, t_0 + \Delta)$.

Definition 10.2 The trivial solution of (10.145) is called stable if for any $\epsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that $|x(t)| < \varepsilon$, for all $t \geq t_0$ if $\|\phi\| = \sup_{\theta\in[t_0-2\Delta,t_0]} |\phi(\theta)| < \delta$.

Definition 10.3 The trivial solution of (10.145) is called locally asymptotically quasistable if it is stable, and if for any $\epsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that the solution of (10.145) is asymptotically quasitrivial for each initial condition (10.146) such that $\|\phi\| < \delta$.

Theorem 10.6 *For a small enough* $\Delta > 0$ *each bounded solution of* (10.145) *with the initial condition* (10.146) *is asymptotically quasitrivial.*

Proof Using the procedure of the construction of the Lyapunov functionals, we will construct a Lyapunov functional for (10.145) in the form $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = \big(x(t) - e^{-\lambda\Delta} x(t - \Delta)\big)^2, \quad t \geq t_0, \tag{10.147}$$

is the Lyapunov functional for the auxiliary linear difference equation (the linear part of (10.145))

$$x(t + \Delta) = x(t) + e^{-\lambda\Delta}\big(x(t) - x(t - \Delta)\big), \quad t > \Delta. \tag{10.148}$$

Indeed, for (10.148) we have $\Delta V_1(t) = 0$. This means that the trivial solution of (10.148) is stable but not asymptotically quasistable.

Calculating $\Delta V_1(t)$ for (10.145) via (10.147), we obtain

$$\begin{aligned} \Delta V_1(t) &= \big(x(t + \Delta) - e^{-\lambda\Delta} x(t)\big)^2 - \big(x(t) - e^{-\lambda\Delta} x(t - \Delta)\big)^2 \\ &= \big(x(t) - e^{-\lambda\Delta} x(t - \Delta) - k\Delta^2 f\big(x(t)\big)\big)^2 - \big(x(t) - e^{-\lambda\Delta} x(t - \Delta)\big)^2 \\ &= k^2\Delta^4 f^2\big(x(t)\big) - 2k\Delta^2 f\big(x(t)\big)x(t) + 2k\Delta^2 e^{-\lambda\Delta} f\big(x(t)\big)x(t - \Delta). \end{aligned} \tag{10.149}$$

Via (10.120) and Lemma 1.1 we have

$$\Delta V_1(t) \le k^2\Delta^4 f^2\big(x(t)\big) - 2k\Delta^2 f\big(x(t)\big)x(t) + 2k\Delta^2 \mathrm{e}^{-\lambda\Delta}\sum_{i=1}^{m}\alpha_i\left(\frac{\nu_i}{\nu_i+1}x^{\nu_i+1}(t) + \frac{1}{\nu_i+1}x^{\nu_i+1}(t-\Delta)\right). \tag{10.150}$$

Put

$$V_2(t) = 2k\Delta^2 \mathrm{e}^{-\lambda\Delta}\sum_{i=1}^{m}\frac{\alpha_i}{\nu_i+1}x^{\nu_i+1}(t-\Delta), \quad t \ge t_0. \tag{10.151}$$

From (10.121) it follows that $V_2(t) \ge 0$. Via (10.150) and (10.151) for the functional $V(t) = V_1(t) + V_2(t)$ we obtain

$$\begin{aligned}\Delta V(t) &\le k^2\Delta^4 f^2\big(x(t)\big) - 2k\Delta^2\big(1-\mathrm{e}^{-\lambda\Delta}\big) f\big(x(t)\big)x(t)\\ &\le -\beta_1(\Delta) f_1\big(x(t)\big)\big(\beta_2(\Delta) - f_2\big(x(t)\big)\big), \quad t \ge t_0,\end{aligned} \tag{10.152}$$

where

$$\begin{aligned}&\beta_1(\Delta) = k^2\Delta^4, \qquad \beta_2(\Delta) = \frac{2(1-\mathrm{e}^{-\lambda\Delta})}{k\Delta^2},\\ &f_1(x) = \sum_{i=1}^{m}\alpha_i x^{\nu_i+1} > 0, \qquad f_2(x) = \sum_{i=1}^{m}\alpha_i x^{\nu_i-1} > 0, \quad x \ne 0.\end{aligned} \tag{10.153}$$

Suppose that there exists $\Delta_0 > 0$, such that the solution of (10.145) is uniformly bounded for $\Delta \in [0, \Delta_0]$, i.e., $|x(t)| \le M$, $t \ge t_0$. Since $f_2(x)$ is a non-decreasing for $x \ge 0$ function and $\lim_{\Delta\to 0}\beta_2(\Delta) = \infty$, then there exists a small enough $\Delta > 0$, such that $f_2(x(t)) \le f_2(M) < \beta_2(\Delta)$. From this and (10.152) it follows

$$\Delta V(t) \le -\gamma_1(\Delta) f_1\big(x(t)\big), \quad t \ge t_0, \tag{10.154}$$

where $\gamma_1(\Delta) = \beta_1(\Delta)(\beta_2(\Delta) - f_2(M)) > 0$. Rewrite (10.154) for $t + j\Delta$, i.e.,

$$\Delta V(t+j\Delta) \le -\gamma_1(\Delta) f_1\big(x(t+j\Delta)\big), \quad t \ge t_0,\ j = 0, 1, \dots,$$

and summing from $j = 0$ to $j = i-1$ we obtain

$$V(t+i\Delta) - V(t) \le -\gamma_1(\Delta)\sum_{j=0}^{i-1} f_1\big(x(t+j\Delta)\big), \quad t \ge t_0. \tag{10.155}$$

From this it follows that

$$\gamma_1(\Delta)\sum_{j=0}^{\infty} f_1\big(x(t+j\Delta)\big) \le V(t) < \infty, \quad t \ge t_0.$$

Therefore, $\lim_{j\to\infty} f_1(x(t+j\Delta)) = 0$ for each $t \ge t_0$. Due to (10.121) and (10.153)

$$0 \le \alpha_1 x^{\nu_1+1}(t+j\Delta) \le f_1\big(x(t+j\Delta)\big), \quad t \ge t_0.$$

So, $\lim_{j\to\infty} |x(t+j\Delta)| = 0$ for each $t \ge t_0$, i.e., the solution of (10.145) is asymptotically quasitrivial. The theorem is proven. □

Theorem 10.7 *The trivial solution of* (10.145) *is stable.*

Proof We will use here the functional $V(t)$ that was constructed in the proof of Theorem 10.6. Via (10.155) we have

$$V(t+i\Delta) \le V(t), \quad i = 0, 1, \dots, \ t \ge t_0.$$

Putting $t = t_0 + j\Delta + s$ with $j = [\frac{t-t_0}{\Delta}]$ and $s \in [0, \Delta)$, we obtain

$$V\big(t_0 + (j+i)\Delta + s\big) \le V(t) = V(t_0 + j\Delta + s) \le V(t_0 + s). \tag{10.156}$$

From (10.147) we have

$$\begin{aligned} V_1(t_0+s) &= \big(x(t_0+s) - \mathrm{e}^{-\lambda\Delta}\phi(t_0+s-\Delta)\big)^2 \\ &\le 2\big(\big|x(t_0+s)\big|^2 + \mathrm{e}^{-2\lambda\Delta}\|\phi\|^2\big). \end{aligned} \tag{10.157}$$

From (10.145) and $t_0 = 2\Delta$ it follows that $t_0 + s - \Delta = \Delta + s \in [\Delta, t_0)$ and

$$\big|x(t_0+s)\big| \le \big(1 + \mathrm{e}^{-\lambda\Delta}\big)\big|\phi(\Delta+s)\big| + a\Delta^2\big|f\big(\phi(\Delta+s)\big)\big| + \mathrm{e}^{-\lambda\Delta}\big|\phi(s)\big|.$$

Due to (10.120)

$$\big|f\big(\phi(\theta)\big)\big| \le \sum_{i=1}^{m} \alpha_i \big|\phi(\theta)\big|^{\nu_i} \le C_2\|\phi\|^{\nu}, \quad \theta \in [0, t_0],$$

where

$$C_2 = \sum_{i=1}^{m} \alpha_i, \qquad \nu = \begin{cases} 1 & \text{if } \|\phi\| \le 1, \\ \max_{i=1,\dots,m} \nu_i & \text{if } \|\phi\| > 1. \end{cases}$$

Therefore,

$$\big|x(t_0+s)\big| \le C_3\|\phi\|^{\nu}, \quad C_3 = 1 + 2\,\mathrm{e}^{-\lambda\Delta} + a\Delta^2 C_2,$$

and using (10.157), we obtain

$$V_1(t_0+s) \le 2\big(C_3^2\|\phi\|^{2\nu} + \mathrm{e}^{-2\lambda\Delta}\|\phi\|^2\big). \tag{10.158}$$

From (10.151) it follows that

$$V_2(t_0+s) = 2k\Delta^2\,\mathrm{e}^{-\lambda\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\nu_i+1}\phi^{\nu_i+1}(t_0+s-\Delta) \le C_1\|\phi\|^{\nu+1}, \tag{10.159}$$

where

$$C_1 = 2k\Delta^2 e^{-\lambda\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\nu_i + 1}.$$

From (10.156), (10.158) and (10.159) for the functional $V(t) = V_1(t) + V_2(t)$ follows the inequality

$$\begin{aligned} &V(t) \le V(t_0 + s) \le C_0 \|\phi\|^{2\nu}, \\ &t \ge t_0, \ C_0 = C_1 + 2\big(C_3^2 + e^{-2\lambda\Delta}\big). \end{aligned} \tag{10.160}$$

Via (10.153), (10.155) and (10.160) we obtain

$$\begin{aligned} \gamma_1(\Delta)\alpha_1 \big|x(t)\big|^{\nu_1+1} &\le \gamma_1(\Delta) f_1\big(x(t)\big) \\ &\le \gamma_1(\Delta) \sum_{j=0}^{i-1} f_1\big(x(t + j\Delta)\big) \le V(t) \le C_0\|\phi\|^{2\nu}, \quad t \ge t_0. \end{aligned}$$

So, for arbitrary $\varepsilon > 0$ there exists a $\delta = (C_0^{-1}\gamma_1(\Delta)\alpha_1\varepsilon^{\nu_1+1})^{\frac{1}{2\nu}} > 0$, such that $|x(t)| < \varepsilon$, if $\|\phi\| < \delta$. The theorem is proven. □

Corollary 10.4 *For a small enough $\Delta > 0$ the trivial solution of* (10.145) *is locally asymptotically quasistable*.

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L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*,
DOI 10.1007/978-0-85729-685-6, © Springer-Verlag London Limited 2011

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